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ASYMPTOTIC GOVERNING EQUATION FOR WAVE PROPAGATION ALONG WEAKLY NON-UNIFORM EULER–BERNOULLI BEAMS

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Non-uniformity in beams arises either from manufacturing imperfections or by design, and can have a singular impact on the qualitative properties of the vibratory response of the beam. To describe the mechanism causing such large changes on the dynamics of the beam, we derived asymptotically a simpler equation, in the form

$$\chi_{ss} + Q(s)\chi(s) = 0.$$

The coefficient function $Q(s)$ is given by equation (52) herein in terms of the beam flexural rigidity, the mass per unit length and the tensile force applied to the beam. The equation is asymptotic to the non-uniformity of the beam, but under certain restrictions, namely of having constant tension and a constant product of the beam mass per unit length and flexural rigidity, it is an exact governing equation for wave propagation along Bernoulli–Euler beams and it has a Helmholtz-like form. The behavior of the equation is systematically explored and illustrated through numerical results.

1. INTRODUCTION

The vibration of non-uniform beams, whether smooth or stepped, has been studied extensively, and is still receiving attention in the literature. The non-uniformity may arise in the material properties and in the geometry of the beam. Usually, the effect of the non-uniformity on the natural frequencies and natural modes of finite beams is investigated, under various boundary conditions. Wave propagation along long non-uniform beams has received less attention, especially in the case of non-periodic, smooth and continuous non-uniformity.

For periodic non-uniformity, analytical techniques can be applied, such as the Floquet theory and perturbation methods, such as the method of multiple scales. Wave interaction with periodic non-uniformity can give rise to strong resonance effects. For incident waves with wavelength of the order of two times the wavelength scale of the non-uniformity, the incident wave resonates with the non-uniformity, leading to almost complete reflection if the non-uniform part of the beam is long enough. This strong interaction is known as Bragg reflection (see reference [1]).

The case of non-periodic non-uniformity is even more interesting. For beams with a long enough non-uniform part, vibration with a fairly broad spectrum may stay localized in a finite region close to its source. The normal modes are not extended anymore, and they

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become localized in space. In terms of wave propagation, we may have almost complete reflection and exponentially small transmission. These are manifestations of localization phenomena in the vibration of mechanical systems. Another feature of the behavior of such localized systems is the disproportionally large sensitivity of the system response with respect to small variation in the non-uniformity of the system parameters, as pointed out by Pierre [2] and Triantafyllou and Tryantafyllou [3].

In summary, small non-uniformity in the material and geometrical properties in beams can have a singular importance, causing large effects on vibration propagation relative to its own magnitude. To help describe these mechanisms well, and given the significance these effects may have for a large number of applications, we derived asymptotically a simpler equation, which captures the essence of the localization phenomena.

We derived asymptotically a second order differential equation governing wave propagation along non-uniform Bernoulli–Euler beams under the action of non-uniform tensile force along their length. This second order governing equation is asymptotic with respect to the steepness of the non-uniformity. In other words, as the non-uniformity steepness decreases, the agreement between the behavior predicted by the second order governing equation and the full governing equation increases.

When the tensile force and the product of the beam flexural rigidity by its mass per unit length are constants, the second order governing equation is an exact governing equation for wave propagation along non-uniform Bernoulli–Euler beams. These restrictions are not commonly met in practice, but such beams can be designed and built.

The second order governing equation under the restrictions mentioned above can be used to design the beam non-uniformity to achieve, for example, passive vibration isolation. This can be done in two ways. The first approach consists of using a second order differential equation with non-constant coefficient which has a closed-form solution and models the wave propagation problem well. The analytical solution furnishes a functional relation between the wave frequency and the scattering coefficients, such as the reflection coefficient, using the non-uniformity in the coefficient function as a parameter. Then, if we specify the form of the functional relation between the reflection coefficient and the wave frequency, we end up designing the non-uniformity in the coefficient function of the second order differential equation. Once the non-uniformity in the coefficient function is specified, the non-uniformity in the beam flexural rigidity and mass per unit length is also determined. We give an example of this approach in section 6.

The second approach consists of using inverse-scattering techniques for Helmholtz-like equations, since the second order governing equation under the restrictions mentioned above has a Helmholtz-like form.

In the next section, we give an outline of the previous work on wave propagation along smooth non-uniform beams, on analytical solutions of second order differential equations and on inverse-scattering techniques for the Helmholtz equation. In section 3, we give the governing equation for the Bernoulli–Euler beam in terms of the chosen non-dimensional variables. In section 4, we derive asymptotically the second order differential equation governing wave propagation along non-uniform Bernoulli–Euler beams. We also discuss the qualitative behavior predicted by this governing equation for general non-uniformities, and we describe the restriction over the non-uniformity under which this equation is an exact governing equation. In section 5, we use the second order governing equation to predict wave reflection by the beam non-uniformity. We compare results for the modulus of the reflection coefficient from the numerical simulation of both governing equations. In section 6, we illustrate the first approach for the design problem mentioned above.

2. PREVIOUS WORK

Wave propagation along smooth non-uniform periodic Bernoulli–Euler beams can be studied through analytical techniques and perturbation methods. Lee and Ke [4] applied the Floquet theory to study flexural wave propagation along Bernoulli–Euler beams with periodic non-uniformity. They show that flexural waves in a periodic beam can be interpreted as a superposition of two pairs of waves propagating in opposite directions, of which one pair behaves as an attenuated wave. Hawwa [1] considered beams with a periodically varying cross-section, and used a straightforward asymptotic expansion to show that resonance between the beam periodicity and the wavefield occur when the wavenumber of the flexural wave is half the wavenumber of that of the beam periodicity. A uniform expansion near the resonance condition was obtained through the method of multiple scales. An account of the literature regarding wave propagation along periodic beams is given in the two references mentioned above.

To study the vibration of non-periodic non-uniform Bernoulli–Euler beams, numerical methods, like the finite element method, the finite difference method and the transfer matrix method are usually applied. The exceptions are geometrical and material non-uniformities which are polynomial functions of the space parametrization. Naguleswaran [5] considered Bernoulli–Euler beams with a variation in breadth proportional to $x^s(s \leqslant 0$, and x is the distance from the "sharp" end). Abrate [6] considered beams with cross-sectional area and its second moment as arbitrary polynomial functions of the space parametrization. Approximations for the natural frequencies were obtained using the Rayleigh–Ritz method.

Heading [7] gives a list of indices of refraction such that the resulting one-dimensional Helmholtz equation has closed-form solutions in terms of transcendental functions. He also discusses how to generate more complex indices of refraction from simple ones. He also gives an account of the literature on one-dimensional Helmholtz- and Schrödinger-like equations with non-constant coefficient functions, which have a closed-form solution.

An account of the literature about inverse-scattering methods for the one-dimensional Helmholtz-like equations up to 1987, with emphasis on the context of seismology, is given in reference [8]. An exact inverse method for the one-dimensional Helmholtz equation in the half line is described in Sylvester *et al.* [9] and Sylvester and Winebrenner [10]. They developed a new layer-stripping technique for the inverse-scattering problem of the Helmholtz equation on the half line. In Sylvester *et al.* [9], they proved convergence of the algorithm and well-posedness of the forward and inverse-scattering problems, and in Sylvester and Winebrenner [10], they constructed a numerical inverse algorithm based on the non-linear Riesz transform.

3. BERNOULLI–EULER BEAM GOVERNING EQUATION

Here we describe the choice of non-dimensional variables and we give the non-dimensional form of the Bernoulli–Euler beam governing equation.

We consider a one-dimensional continuous model for an elastic beam under small transverse motions, which are represented by the transversal displacement $\eta(x, t)$ of the neutral line of the beam cross-section. The variables x and t represent, respectively, the space and time co-ordinates. The Bernoulli–Euler beam model assigns only transverse inertia (i.e., ignores rotational inertia) and bending elasticity (i.e., ignores shear deformation) to the continuum. We also consider the elastic restoring force due to the tensile force applied to the beam. For the beam model considered, the governing equation for the transversal displacement results from the balance between the cross-sectional inertia force and the gradient of the shear force due to the bending moment and tensile force, as follows:

$$(I(x)E(x)\eta_{xx})_{xx} - (P(x)\eta_x)_x + \rho(x)A(x)\eta_{tt} = 0. \tag{1}$$

$A(x)$ and $I(x)$ are, respectively, the cross-sectional area and its second moment. $E(x)$ and $\rho(x)$ are, respectively, the modulus of elasticity and density of the material. $P(x)$ represents the tensile load applied to the beam. The Bernoulli–Euler beam is considered non-uniform for $0 \leqslant x \leqslant L$. For $x \leqslant 0$ and $x \geqslant L$, the beam is uniform, but not necessarily with the same geometrical and material properties. We consider the following set of non-dimensional variables:

$$s = \frac{x}{\lambda}, \qquad \tau = f_0 t \quad \text{with} \quad f_0 = \sqrt{\frac{E_0 I_0}{\rho_0 A_0 \lambda^4}},$$
$$\omega = \frac{f_0}{\Omega}, \qquad y = \frac{\eta}{h_0}, \qquad \bar{L} = \frac{L}{\lambda}, \tag{2}$$
$$\bar{P}(s) = \frac{\lambda^2}{E_0 I_0} P(x), \qquad m(s) = \frac{\rho(x)A(x)}{\rho_0 A_0}, \qquad ei(s) = \frac{I(x)E(x)}{I_0 E_0},$$

where λ is the wavelength of the wave disturbance on the uniform part of the beam at $x \leqslant 0$. The quantities $\rho_0 A_0$ and $E_0 I_0$ are, respectively, the reference values for the mass per unit length and for the flexural rigidity. The quantity h_0 is the half-beam-cross-section reference height, and f_0 is the time non-dimensionalization factor. All reference quantities are taken from the uniform part located at $x \leqslant 0$. In terms of the non-dimensional variables, the governing equation (1) assumes the form

$$(ei(s)y_{ss})_{ss} - (\bar{P}(s)y_s)_s + m(s)y_{\tau\tau} = 0. \tag{3}$$

The non-dimensional mass per unit length $m(s)$ and the non-dimensional flexural rigidity $ei(s)$ are defined as follows:

$$m(s) = \begin{cases} 1 & \text{for } s < 0, \\ \dfrac{\rho A(\lambda s)}{\rho_0 A_0} & \text{for } 0 \leqslant s \leqslant \bar{L}, \\ \dfrac{\rho_1 A_1}{\rho_0 A_0} & \text{for } s > \bar{L}, \end{cases}$$

and (4)

$$ei(s) = \begin{cases} 1 & \text{for } s < 0, \\ \dfrac{EI(\lambda s)}{E_0 I_0} & \text{for } 0 \leqslant s \leqslant \bar{L}, \\ \dfrac{E_1 I_1}{E_0 I_0} & \text{for } s > \bar{L}. \end{cases}$$

Since we are interested in the interaction of mono-chromatic waves with the beam non-uniformity, we assume the time dependence

$$\exp(-\mathrm{i}\omega\tau), \tag{5}$$

with ω as the non-dimensional wave frequency. The non-dimensional governing equation (3) assumes the form

$$(ei(s)\,y_{ss})_{ss} - (\bar{P}(s)\,y_s)_s - \omega^2 m(s)\,y(s) = 0, \tag{6}$$

which is a fourth order differential equation with variable coefficients.

4. SECOND ORDER GOVERNING EQUATION FOR THE BERNOULLI–EULER BEAM

In this section, we derive asymptotically a second order differential equation governing wave propagation along a non-uniform Bernoulli–Euler beam.

First, we consider a change of the dependent variable to transform the governing equation (6) into a four-dimensional system of first order differential equations. Second, we discuss the restrictions on the beam non-uniformity which allows wave propagation to be governed by a two-dimensional system of first order differential equations, and from this system of equations we obtain a second order differential equation through a sequence of changes of the dependent variable. We also discuss the qualitative insight of the effects of the non-uniformity on wave propagation given by the second order governing equation. Third, we pay special attention to beams where the tensile force and the product of the beam flexural rigidity by its mass per unit length are constants. When these restrictions are satisfied, the second order governing equation is an exact governing equation, and has a Helmholtz-like form.

4.1. GOVERNING EQUATION FOR THE WAVE MODES AMPLITUDE

Here we consider a change of the dependent variable which transforms equation (6) in a four-dimensional system of first order differential equations.

Along the uniform part of the beam, the general solution of the governing equation (6) is given as a superposition of four wave modes, as follows:

$$y(s) = A\exp(\mathrm{i}\,k_1 s) + B\exp(-\mathrm{i}\,k_1 s) + C\exp(k_2 s) + D\exp(-k_2 s). \tag{7}$$

In equation (7), the first two wave modes are propagating modes, and the last two are evanescent modes. Each wave mode has an associated wavenumber. For the propagating modes, the wavenumbers are pure imaginary numbers with the same modulus, but opposite phase. For the evanescent modes, the wavenumbers are real numbers with the same modulus, but opposite sign. The wavenumbers $\mathrm{i}\,k_1$ and k_2 are solutions of the dispersion relation

$$ei\,k^4 - \bar{P}k^2 - \omega^2 m = 0. \tag{8}$$

The wave mode amplitudes A, B, C and D are specified by boundary conditions (finite beam) or by radiation conditions.

The rotation of the beam cross-section, the bending moment and the shear force can also be written in terms of the wave modes.

We assume that the transverse displacement along the non-uniform part of the beam can be written in the same form as in the case of the uniform beam, but now with the wave mode amplitudes and wavenumbers as functions of the space co-ordinate, as follows:

$$y(s) = \underbrace{A(s)\exp(\mathrm{i}\,k_1(s)s)}_{\tilde{A}(s)} + \underbrace{B(s)\exp(-\mathrm{i}\,k_1(s)s)}_{\tilde{B}(s)} + \underbrace{C(s)\exp(k_2(s)s)}_{\tilde{C}(s)}$$

$$+ \underbrace{D(s)\exp(-k_2(s)s)}_{\tilde{D}(s)}. \tag{9}$$

The new dependent variables are defined by incorporating the phase factor to the wave mode amplitudes. These new dependent variables are denoted as $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$, according to equation (9). The wavenumbers are still given by the uniform system dispersion relation (8), now assumed locally valid and with the non-dimensional flexural rigidity ei, the non-dimensional mass per unit length m and the non-dimensional tensile force $\bar{P}$ as functions of the space co-ordinate. The wavenumbers are now given by the equations

$$\mathrm{i}\,k_1(s) = \frac{\mathrm{i}}{\sqrt{2ei(s)}}\{-\bar{P}(s) + \sqrt{\bar{P}^2(s) + 4\omega^2 ei(s)m(s)}\}^{1/2}, \tag{10}$$

$$k_2(s) = \frac{1}{\sqrt{2ei(s)}}\{\bar{P}(s) + \sqrt{\bar{P}^2(s) + 4\omega^2 ei(s)m(s)}\}^{1/2}. \tag{11}$$

For the non-uniform beam, we assume that the rotation of the beam cross-section, the bending moment and the shear force along the non-uniform beam are given in the same form as in the case of the uniform beam, but now with wave mode amplitudes and wavenumbers as functions of the space co-ordinate. The functions $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$ will be sought such that

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \mathrm{i}\,k_1(s)(\tilde{A}(s) - \tilde{B}(s)) + k_2(s)(\tilde{C}(s) - \tilde{D}(s)), \tag{12}$$

$$ei(s)\frac{\mathrm{d}^2y}{\mathrm{d}s^2} = -k_1^2(s)(\tilde{A}(s) + \tilde{B}(s)) + k_2^2(s)(\tilde{C}(s) + \tilde{D}(s)), \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(ei(s)\frac{\mathrm{d}^2y}{\mathrm{d}s^2}\right) - \bar{P}(s)\frac{\mathrm{d}y}{\mathrm{d}s} = -\mathrm{i}\,k_1(s)(k_1^2(s) + ei(s)[k_2^2(s) - k_1^2(s)])(\tilde{A}(s) - \tilde{B}(s))$$

$$+ k_2(s)(k_2^2(s) - ei(s)[k_2^2(s) - k_1^2(s)])(\tilde{C}(s) - \tilde{D}(s)). \tag{14}$$

These representations for the rotation of the beam cross-section, for the bending moment and for the shear force can be justified if and only if equations (A.2)–(A.4) are satisfied (see Appendix A). This set of equations furnishes the three first order differential equations for the quantities $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$. If we substitute the equations for the transverse displacement (9), for the bending moment (13) and for the shear stress (14) on the governing equation (6), we obtain one more equation involving the quantities $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$ and their first order derivatives, which is equation (A.5) in Appendix A.

Equation (A.5) plus equations (A.2)–(A.4) furnish a four-dimensional system of first order differential equations for the new dependent variables $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$, as follows:

$$\begin{Bmatrix} \dfrac{\mathrm{d}\tilde{A}(s)}{\mathrm{d}s} \\ \dfrac{\mathrm{d}\tilde{B}(s)}{\mathrm{d}s} \\ \dfrac{\mathrm{d}\tilde{C}(s)}{\mathrm{d}s} \\ \dfrac{\mathrm{d}\tilde{D}(s)}{\mathrm{d}s} \end{Bmatrix} = [\mathbf{M}(s)] \begin{Bmatrix} \tilde{A}(s) \\ \tilde{B}(s) \\ \tilde{C}(s) \\ \tilde{D}(s) \end{Bmatrix}. \tag{15}$$

The equations describing the elements of the system matrix, denoted $\mathbf{M}(s)$, in terms of the system parameters are given in Appendix A.

The linear combination of the quantities $\tilde{A}(s)$ and $\tilde{B}(s)$ accounts for the propagating component of the wave disturbance along the non-uniform beam, and the linear combination of the quantities $\tilde{C}(s)$ and $\tilde{D}(s)$ accounts for the evanescent component of the wavefield along the non-uniform beam. Radiation conditions for left or right incidence can be written entirely in terms of the quantities $\tilde{A}(s)$ and $\tilde{B}(s)$. Therefore, under conditions of weak or no coupling between the quantities $\tilde{A}(s)$ and $\tilde{B}(s)$ and the quantities $\tilde{C}(s)$ and $\tilde{D}(s)$, wave propagation along the non-uniform beam should be entirely described by the evolution of the quantities $\tilde{A}(s)$ and $\tilde{B}(s)$, as is shown in the next section.

4.2. ASYMPTOTIC SECOND ORDER GOVERNING EQUATION

We define quantities that are useful in discussing the restrictions necessary for the evolution of the quantities $\tilde{A}(s)$ and $\tilde{B}(s)$ to decouple asymptotically from the evolution of the quantities $\tilde{C}(s)$ and $\tilde{D}(s)$.

The ratio $v(s)$ between the inertia force per unit length and the flexural rigidity and the ratio $\theta(s)$ between the tensile load and the flexural rigidity are defined, respectively, as

$$v(s) = \frac{\omega^2 m(s)}{ei(s)}, \qquad \theta(s) = \frac{\bar{P}(s)}{ei(s)}. \tag{16, 17}$$

We can express the wavenumbers $k_1(s)$ and $k_2(s)$ and their derivative in terms of the quantities defined above, according to equations (B.1)–(B.4) of Appendix B. The derivatives of the quantities $v(s)$ and $\theta(s)$, present in equations (B.3) and (B.4) for the derivatives of the wavenumbers, are given in terms of the beam parameters, as follows:

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{v(s)}{m(s)}\frac{\mathrm{d}m}{\mathrm{d}s} - \frac{v(s)}{ei(s)}\frac{\mathrm{d}\,ei}{\mathrm{d}s}, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{ei(s)}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s} - \frac{\theta(s)}{ei(s)}\frac{\mathrm{d}\,ei}{\mathrm{d}s}. \tag{18, 19}$$

The equations for elements of matrix $\mathbf{M}(s)$ are given in Appendix A in terms of the beam parameters, their derivatives, and in terms of the wavenumbers $k_1(s)$ and $k_2(s)$ and their derivatives. Therefore, we need to estimate the order of magnitude of these quantities to be able to study the order of magnitude of the matrix elements $\mathbf{M}_{jk}$. The non-dimensional mass per unit length and the non-dimensional flexural rigidity are considered as quantities of order $O(1)$. To estimate the order of magnitude of the derivatives of these quantities, we define the ratio between the length scale λ of the incident wave and the length scale λ' of the non-uniformity variation as

$$\Lambda = \frac{\lambda}{\lambda'}. \tag{20}$$

The magnitude of the non-uniformity in the non-dimensional flexural rigidity, in the non-dimensional mass per unit length and in the non-dimensional tensile force per one wavelength of the incident wave are defined as

$$\max_{0<s<\bar{L}} \{|ei(s+1) - ei(s)|\} \sim O(\varepsilon'), \tag{21}$$

$$\max_{0<s<\bar{L}} \{|m(s+1) - m(s)|\} \sim O(\varepsilon'), \tag{22}$$

$$\max_{0<s<\bar{L}} \{|\bar{P}(s+1) - \bar{P}(\mathrm{s})|\} \sim O(\delta). \tag{23}$$

Now, we can write the magnitude of the derivatives of the non-dimensional tensile force, of the non-dimensional mass per unit length and of the non-dimensional flexural rigidity in terms of the quantities defined above, as follows:

$$\frac{\mathrm{d}ei}{\mathrm{d}s} \sim O(\varepsilon'\Lambda), \qquad \frac{\mathrm{d}m}{\mathrm{d}s} \sim \mathrm{O}(\varepsilon\Lambda), \qquad \frac{\mathrm{d}\bar{P}}{\mathrm{d}s} \sim O(\delta\Lambda). \tag{24–26}$$

The ratio between the non-dimensional inertia force per unit length and the non-dimensional flexural rigidity is a quantity of order $O(1)$ for large values of the frequency of the wave disturbance. The exception is the long wave limit, where $v(s)$ is a quantity of order of magnitude larger than one. In general, we can say that

$$v(s) \sim O(1). \tag{27}$$

Based on the estimates above for the order of magnitude of the non-dimensional mass per unit length, non-dimensional flexural rigidity, non-dimensional tensile force and their derivatives, we are ready to estimate the order of magnitude of the elements of matrix $\mathbf{M}(s)$. Since they can all be expressed, basically, in terms of the elements $M_{11}(s)$, $M_{21}(s)$, $M_{31}(s)$, $M_{13}(\mathrm{s})$, $M_{33}(s)$, $M_{43}(s)$ and M_{44}, these are the only ones which need to be analyzed. We denote them as the basic elements of matrix $\mathbf{M}(s)$.

We consider two regimes. First, the non-dimensional tensile force is very small compared to the non-dimensional flexural rigidity ($\theta \to 0$). Second, the non-dimensional tensile force is of the same or larger order than the order of magnitude of the non-dimensional flexural rigidity ($\theta \geqslant 1$).

For each of the two regimes considered, we express in Appendix B the order of magnitude of the wavenumbers and their derivatives. We also describe in detail the order of magnitude of the basic elements of the matrix $\mathbf{M}(s)$.

4.2.1. *Regime $\theta \to 0$*

The order of magnitude of the basic elements of the matrix $\mathbf{M}(s)$ is given in Appendix B.1 in terms of the order of magnitude of the beam parameters and their derivatives, according to equations (B.10)–(B.16). We define ε'' as

$$\varepsilon'' = \max\{\varepsilon, \varepsilon', \delta\}. \tag{28}$$

Now, the order of magnitude of the basic elements of matrix $\mathbf{M}(s)$ is given as

$$M_{11}(s) \sim O(v^{1/4}), \qquad M_{21}(s) \sim O(\varepsilon''\Lambda), \tag{29, 30}$$

$$M_{31}(s) \sim O(\varepsilon''\Lambda), \qquad M_{13}(s) \sim O(\varepsilon''\Lambda), \tag{31, 32}$$

$$M_{33}(s) \sim O(v^{1/4}), \qquad M_{43}(s) \sim O(\varepsilon''\Lambda), \qquad M_{44}(s) \sim O(\varepsilon''\Lambda). \tag{33–35}$$

We assume the non-uniformity steepness of the beam parameters to be small. In other words,

$$\varepsilon''\Lambda \ll 1. \tag{36}$$

According to equations (29)–(35) and under the restriction given by equation (36), the evolution described by the four-dimensional system of first order differential equations (15) can be asymptotically described by the evolution of a diagonal system of first order differential equations with an error of $O(\varepsilon''\Lambda)$, which is small. This diagonal matrix is just the main diagonal of matrix $\mathbf{M}(s)$. Therefore, under this regime and the restriction given above, the evolution of the components describing the propagating and evanescent parts of the solution can be considered decoupled with an error of the order of $O(\varepsilon''\Lambda)$. Since we want to study resonance effects between the non-uniformity and the wavefield, we keep the coupling between the components describing the propagating part of the wavefield. Therefore, the propagating part of the solution is described with an error of the order of $O(\varepsilon''\Lambda)$ by a two-dimensional system of first order differential equations with matrix given by the 2×2 upper left block of matrix $\mathbf{M}(s)$.

4.2.2. *Regime $\theta \geqslant 1$*

The order of magnitude of the basic elements of the matrix $\mathbf{M}(s)$ is given in Appendix B.2 in terms of the order of magnitude of the beam parameters and their derivatives according to equations (B.22)–(B.28).

We consider the same assumption as in the previous regime regarding the order of magnitude of the steepness of the non-uniformity. We give the order of magnitude of the basic elements of the matrix $\mathbf{M}(s)$, as follows:

$$M_{11}(s) \sim O(v^{1/4}), \qquad M_{21}(s) \sim O(v^{-1/2}\theta\varepsilon''\Lambda), \tag{37, 38}$$

$$M_{31}(s) \sim O(\varepsilon''\Lambda), \qquad M_{13}(s) \sim O(v^{1/4}\theta^{1/2}\varepsilon''\Lambda), \tag{39, 40}$$

$$M_{33}(s) \sim O(\theta^{1/2}), \qquad M_{43}(s) \sim O(\varepsilon''\Lambda), \qquad M_{44}(s) \sim O(\theta^{1/2}). \tag{41–43}$$

According to equations (37)–(43), even with the assumption of small steepness, the coupling between the propagation part and the evanescent part of the solution is not necessarily small. For the propagating part of the solution to be described by a two-dimensional system of first order differential equations, we need little energy to be transferred from the propagating part of the solution to its evanescent part. The transfer of energy from the propagating modes to the evanescent modes is governed by the elements $M_{31}(s)$, $M_{32}(s)$, $M_{41}(s)$, and $M_{42}(s)$, which have an order of magnitude of the order of $O(\varepsilon''\Lambda)$. Therefore, under the restriction given by equation (36), the transfer of energy between the propagating and evanescent components of the wavefield is small, and the evolution of the propagating part of the wavefield can be described by the two-dimensional system of first order differential equation with matrix given by the 2×2 upper left block of matrix $\mathbf{M}(s)$. The error in this approximation is of the order of $O(\varepsilon''\Lambda)$, which is small.

In both regimes, if the restriction of small steepness in the non-uniformity is satisfied, the propagating part of the wavefield can be described asymptotically by a two-dimensional system of first order differential equations with matrix given by the 2×2 upper left block of matrix $\mathbf{M}(s)$, as follows:

$$\left\{\begin{matrix} \dfrac{\mathrm{d}\tilde{A}(s)}{\mathrm{d}s} \\ \dfrac{\mathrm{d}\tilde{B}(s)}{\mathrm{d}s} \end{matrix}\right\} = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} \left\{\begin{matrix} \tilde{A}(s) \\ \tilde{B}(s) \end{matrix}\right\}. \tag{44}$$

This two-dimensional system of equations can be reduced through a change of the dependent variables to a second order differential equation, as discussed in the next section.

4.2.3. *Second order differential equation*

We describe the change of the dependent variables used to obtain the second order governing equation from the system of equation (44), which governs the evolution of the propagating part of the wavefield.

The new dependent variables are $\psi(s)$, the transverse displacement, and $\phi(s)$, the rotation of the beam cross-section due to the propagating component of the wavefield, normalized by $-k_1(s)$. These new dependent variables are related to the quantities $\tilde{A}(s)$ and $\tilde{B}(s)$ through the matrix equation

$$\left\{\begin{matrix} \psi(s) \\ \phi(s) \end{matrix}\right\} = \begin{bmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{bmatrix} \left\{\begin{matrix} \tilde{A}(s) \\ \tilde{B}(s) \end{matrix}\right\}. \tag{45}$$

We substitute the matrix equation (45) into the system of equations (44). This leads to a new system of equations in terms of the new dependent variables ψ and ϕ.

The first equation of the system of equations for the dependent variables ψ and ϕ allows us to write the variable ϕ in terms of the transverse displacement ψ and its derivative $\mathrm{d}\psi/\mathrm{d}s$. If we substitute this expression for ϕ into the second equation of the system of equations for the dependent variables ψ and ϕ, we obtain the second order differential equation

$$\psi_{ss} + p(s)\psi_s + q(s)\psi(s) = 0. \tag{46}$$

The quantities $p(s)$ and $q(s)$ are defined in terms of the beam parameters and in terms of the wavenumber $k_1(s)$ and $k_2(s)$ as

$$p(s) = \frac{1}{ei(s)}\frac{\mathrm{d}ei(s)}{\mathrm{d}s} + \frac{1}{k_1^2(s) + k_2^2(s)}\frac{\mathrm{d}}{\mathrm{d}s}(k_1^2(s) + k_2^2(s)), \tag{47}$$

$$q(s) = \frac{1}{4}\left(\frac{1}{ei(s)}\frac{\mathrm{d}ei(s)}{\mathrm{d}s}\right)^2 - \frac{1}{[k_1^2(s) + k_2^2(s)]^2}\left[2\,k_1(s)\frac{\mathrm{d}k_1}{\mathrm{d}s} - \frac{\bar{P}}{2(ei(s))^2}\frac{\mathrm{d}ei}{\mathrm{d}s}\right]^2 + \frac{1}{k_1^2(s) + k_2^2(s)}\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left[2k_1(s)\frac{\mathrm{d}k_1}{\mathrm{d}s} + \frac{k_1^2(s)}{ei(s)}\frac{\mathrm{d}ei}{\mathrm{d}s}\right]\right\} + k_1^2(s). \tag{48}$$

Let us write this second order equation in a more standard form through a change of the dependent variable, which is given below by the equation

$$\psi(s) = \chi(s)\exp\left(-\frac{1}{2}\int^s p(t)\,\mathrm{d}t\right) = \chi(s)\left\{\bar{P}^2(s) + 4\omega^2 m(s)\,ei(s)\right\}^{-1/4}. \tag{49}$$

Equation (46) reduces to the form

$$\chi_{ss} + Q(s)\chi(s) = 0. \tag{50}$$

The coefficient function $Q(s)$ is given in terms of $p(s)$ and $q(s)$ by the equation

$$Q(s) = -\frac{1}{2}\frac{\mathrm{d}p}{\mathrm{d}s} - \frac{1}{4}(p(s))^2 + q(s). \tag{51}$$

In terms of the beam non-dimensional flexural rigidity, non-dimensional tensile force and non-dimensional mass per unit length, the coefficient function $Q(s)$ assumes the form

$$Q(s) = -\frac{1}{2}F(s)^{-1/2}\frac{\mathrm{d}^2\bar{P}}{\mathrm{d}s^2} - \frac{1}{4F(s)}\left(\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}\right)^2 + \frac{1}{4}F(s)^{-3/2}\frac{\mathrm{d}F}{\mathrm{d}s}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s} + k_1^2(s), \tag{52}$$

with

$$F(s) = \bar{P}(s)^2 + 4\omega^2 m(s)ei(s). \tag{53}$$

Before discussing the aspects of the qualitative behavior of the second order differential equation (49), we discuss the radiation conditions for left and right wave incidence. The radiation conditions are imposed on the uniform parts of the beam ($s \to \pm\infty$). In terms of the variable $\psi(s)$, the radiation conditions for left and right wave incidence follows.

- For left wave incidence we have the radiation conditions

$$\psi(s) = \begin{cases} A\exp(\mathrm{i}k_1^- s) + R^- A\exp(-\mathrm{i}k_1^- s) & \text{as } s \to -\infty, \\ T^- A\exp(\mathrm{i}k_1^+ s) & \text{as } s \to +\infty. \end{cases} \tag{54}$$

- For right wave incidence we have the radiation conditions

$$\psi(s) = \begin{cases} T^+ A\exp(-\mathrm{i}k_1^- s) & \text{as } s \to -\infty, \\ A\exp(-\mathrm{i}k_1^+ s) + R^+ A\exp(\mathrm{i}k_1^+ s) & \text{as } s \to +\infty. \end{cases} \tag{55}$$

A is the amplitude of the incident wave, and R^- and T^- (R^+ and T^+) are the reflection and transmission coefficients for left (right) wave incidence radiation condition. The wavenumbers k_1^- and k_1^+ are the values of the wavenumber $k_1(s)$ for $s \leqslant 0$ and $s \geqslant \bar{L}$ respectively.

Along the uniform part of the beam $\chi(s)$ is proportional to $\psi(s)$. Therefore, the radiation conditions in terms of the dependent variable $\chi(s)$ has the same form as the radiation condition in terms of the dependent variable $\psi(s)$. If the parameters of the beam are the same at both semi-infinite uniform parts, the radiation conditions in terms of the variables ψ and χ are exactly the same. The radiation condition in terms of the variable $\chi(s)$ follows.

- For left wave incidence we have the radiation conditions

$$\chi(s) = \begin{cases} A\exp(\mathrm{i}k_1^- s) + \tilde{R}^- A\exp(-\mathrm{i}k_1^- s) & \text{as } s \to -\infty, \\ \tilde{T}^- A\exp(\mathrm{i}k_1^+ s) & \text{as } s \to +\infty. \end{cases} \tag{56}$$

- For right wave incidence we have the radiation conditions

$$\chi(s) = \begin{cases} \tilde{T}^+ A\exp(-\mathrm{i}k_1^- s) & \text{as } s \to -\infty, \\ A\exp(-\mathrm{i}k_1^+ s) + \tilde{R}^+ A\exp(\mathrm{i}k_1^+ s) & \text{as } s \to +\infty. \end{cases} \tag{57}$$

$\tilde{R}^-$ and $\tilde{T}^-$ ($\tilde{R}^+$ and $\tilde{T}^+$) are the reflection and transmission coefficients for left (right) wave incidence in terms of the variable $\chi(s)$. Between the coefficients $T^\pm$ and $\tilde{T}^\pm$ we have the following relation:

$$\tilde{T}^- = \left\{\frac{(P_0)^2 + 4\omega^2 m_0\, ei_0}{(P_1)^2 + 4\omega^2 m_1\, ei_1}\right\}^{1/4} T^-, \tag{58}$$

$$\tilde{T}^+ = \left\{\frac{(P_1)^2 + 4\omega^2 m_1\, ei_1}{(P_0)^2 + 4\omega^2 m_0\, ei_0}\right\}^{1/4} T^+. \tag{59}$$

In equations (58) and (59), the indices 0 and 1 in the beam parameters refer, respectively, to the parameter values for $s \leqslant 0$ and $s \geqslant \bar{L}$. The reflection coefficients $R^\pm$ are equal to the reflection coefficients $\tilde{R}^\pm$.

In the next section, we discuss the qualitative behavior of the second order differential equation in terms of the Bernoulli–Euler beam parameters.

4.2.4. *Properties of the second order differential equation*

The second order differential equation (49) in terms of the variable $\chi(s)$ satisfies energy flux conservation. In other words,

$$|\tilde{R}^\pm|^2 + |\tilde{T}^\pm|^2 = 1. \tag{60}$$

This is not true for the second order differential equation (46). Its reflection and transmission coefficients satisfy the relation

$$\pm\, \mathrm{i}2k_1 A^2 (\pm |T^\mp|^2 \pm |R^\mp|^2 \mp 1) = \exp\left(-\int_0^{\bar{L}} p(t)\,\mathrm{d}t\right). \tag{61}$$

The coefficient function $Q(s)$ can be simplified if we make assumptions regarding the Bernoulli–Euler beam parameters. If these parameters are constants, the coefficient function $Q(s)$ is a constant, which is equal to the square of the wavenumber k_1. The second order equation (50) in this case is a one-dimensional Helmholtz equation with constant wavenumber k_1, which is the wavenumber associated with the propagating modes for the Bernoulli–Euler beam with constant material and geometrical properties. Therefore, along the uniform part of the beam, the coefficient function $Q(s)$ gives the right wavenumber for the propagating modes.

If we assume only the tensile force to be a constant, equation (52) simplifies to

$$Q(s) = k_1^2(s). \tag{62}$$

The second order differential equation under the restriction of constant tensile force looks like a Helmholtz equation. This is not true, since we cannot write the coefficient function $Q(s)$ as a "wave frequency" times an index of refraction. The dependence on the frequency for the coefficient function $Q(s)$ in the above form is more complicated than the dependence encountered in Helmholtz-like equations, as revealed by equation (10). If we impose special restrictions on the non-dimensional mass per unit length and on the non-dimensional flexural rigidity, we end up with a Helmholtz-like equation, which is discussed in the next section.

Since the asymptotic governing equation is a second order differential equation, its qualitative behavior with respect to the non-dimensional tensile force, non-dimensional mass per unit length and non-dimensional flexural rigidity is revealed through the study of the sign of the coefficient function $Q(s)$. When the non-dimensional tensile is a constant, the coefficient function $Q(s)$ is always positive, and the governing equation predicts propagating wave-like solutions. If the tensile force is not a constant, the coefficient function $Q(s)$ (see equation (52)) may assume negative values, which implies exponentially decaying or growing solutions. If this happens for some interval of s along the beam, we have exponentially small wave transmission and almost complete wave reflection. If $Q(s)$ happens to be negative for a range of wave frequencies at disjoint intervals of the space co-ordinate s, trapped modes for some wave frequency values may exist.

The coefficient function $Q(s)$ may be negative for some range of s, according to equation (52), if and only if

$$\frac{\mathrm{d}^2\bar{P}}{\mathrm{d}s^2} > 0, \qquad \frac{\mathrm{d}F}{\mathrm{d}s}\frac{\mathrm{d}\bar{P}(s)}{\mathrm{d}s} < 0. \tag{63, 64}$$

The full governing equation should behave in the way predicted by the second order governing equation (49), specially if the restriction of small non-uniformity steepness is satisfied. Therefore, a Bernoulli–Euler beam under variable tensile force satisfying equations (63) and (64) may have trapped modes, and allow only exponentially small transmission. In section 5, we give an example of a non-uniformity which satisfies equations (63) and (64) for intervals of the space co-ordinate s.

4.3. SECOND ORDER GOVERNING EQUATIONS AS AN EXACT GOVERNING EQUATION

According to the system of equations (15), the coupling between the propagating and evanescent part of the wavefield is governed by the elements $M_{jk}(s)$ and $M_{kj}(s)$ ($j = 1, 2$ and $k = 3, 4$). If the non-dimensional tensile force and the product of the non-dimensional mass per unit length by the non-dimensional flexural rigidity are constants, we show in Appendix C that the matrix elements $M_{jk}(s)$ and $M_{kj}(s)$ are zero, which implies that the second order differential equation (49) is an exact equation governing monochromatic wave propagation along the non-uniform beam, and assumes the Helmholtz-like form

$$\chi_{ss} + \hat{\Omega}^2(\omega, \bar{P}, C)\tilde{n}^2(s)\chi(s) = 0. \tag{65}$$

$\tilde{n}(s)$ is the index of refraction, which is given in terms of the non-dimensional flexural rigidity or non-dimensional mass per unit length, according to the equation

$$\tilde{n}(s) = \frac{1}{\sqrt{ei(s)}} = \sqrt{\frac{m(s)}{C}}. \tag{66}$$

The "wave frequency" $\hat{\Omega}(\omega, \bar{P}, C)$ is defined by the equation

$$\hat{\Omega}(\omega, \bar{P}, C) = \left\{ -\frac{\bar{P}}{2} + \frac{1}{2}\sqrt{\bar{P}^2 + 4\omega^2 C} \right\}^{1/2}. \tag{67}$$

We define the dimensional counterpart of $\chi(s)$ as $w(x)$. The relation between these two dependent variables is given by the equation

$$w(x) = \frac{h_0\sqrt{E_0 I_0}}{\lambda}\chi(s). \tag{68}$$

The square of the variable $w(x)$ has the dimension of force times length squared. The dimensional form of the second order governing equation (65) follows:

$$w_{xx} + \tilde{\Omega}^2(P, \Omega, C')n(x)^2 w(x) = 0, \tag{69}$$

where the index of refraction $n(x)$ is given by the equation

$$n(x) = \frac{1}{\sqrt{EI(x)}} = \sqrt{\frac{\rho A(x)}{C'}}. \tag{70}$$

The "wave frequency" $\tilde{\Omega}(P, \Omega, C')$ has the dimension of force and its expression in terms of the constant tensile force P and the constant C' is given by the equation

$$\tilde{\Omega}(P, \Omega, C') = \left\{ -\frac{P}{2} + \frac{1}{2}\sqrt{P^2(x) + 4\Omega\, C'} \right\}^{1/2}. \tag{71}$$

The constant C' has the dimension of mass times energy, and it is the value of the product of the mass per unit length by the flexural rigidity along the entire beam.

5. APPLICATIONS—THE ANALYSIS PROBLEM

In this section, we consider wave interaction with five examples of non-uniformities. For the first three examples, the beam material properties are assumed to be constant. We allow variations only in the cross-sectional geometry and in the tensile force. For the last two examples, we prescribe the flexural rigidity and the mass per unit length such that their product is a constant. The tensile force is assumed constant.

For each example of non-uniformity, we give results for the modulus of the reflection coefficient for both governing equations. Results from the second and full governing equations are obtained through numerical simulation. The exception is the last example of non-uniformity, where the second order governing equation has an analytic solution.

We use the finite difference method to simulate numerically the governing equations. The radiation conditions are incorporated in the finite difference method.

A detailed description of each example considered is given in Table 1.

5.1. NON-UNIFORM GEOMETRY

For the first three examples, we considered a beam with rectangular cross-section. The only non-uniformity is in the changing dimensions of the cross-section. The material properties are considered constant along the entire beam, which is assumed to be made of aluminium with density $\rho_0 = 2{\cdot}7 \times 10^3\ \mathrm{kg/m^3}$ and with modulus of elasticity $E_0 = 7{\cdot}1 \times 10^{10}\ \mathrm{N/m^2}$. Along the non-uniform part of the beam, the height of the cross-section varies in a prescribed way. The heights below and above the mean line are prescribed by different given functions. Along the uniform parts of the beam ($s \leqslant 0$ and $s \geqslant \bar{L}$), the height h_0 of the cross-section is constant and its value is 0·01 m. The width of the cross-section is constant along the entire beam, and it is denoted as b. The value of the width b is 0·05 m.

The non-dimensionalization constants for the mass per unit length and flexural rigidity in the first three examples are $\rho_0 A_0 = \rho_0 b h_0$ and $E_0 I_0 = E_0\,(b h_0^3/12)$, and the non-dimensional expressions for the mass per unit length and flexural rigidity in terms of the functions $f(s)$

Table 1

Description of the examples

Example / Associated figure	First / Figure 1	Second / Figure 2	Third / Figure 3	Fourth / Figure 4	Fifth / Figure 5
Non-dimensional flexural rigidity		Equation (73)		Equation (77)	Equation (78)
Non-dimensional mass per unit length		Equation (72)		$m(s)ei(s) = C$	
Non-dimensional tensile force	constant		Equation (76)	constant	
Function $f(s)$	Equation (74)		Equation (75)		
Function $g(s)$	$= f(s)$	$= -f(s)$	$= -f(s)$		
Constant C				1	1
Non-uniformity parameters	A/h_0	A/h_0	$\frac{A}{h_0}$, A_p, k_g and k_p	A, k_g, s_0, γ and κ	S, N, M and s_0
Parameter A/h_0	0·2, 0·1 and 0·05	0·1, 0·05 and 0·01	0·05 and 0·01		
Parameter A				0·2, 0·1 and 0·05	
Parameter A_P			0 N and 10 000 N		
For other parameters			Caption of Figure 3	Caption of Figure 4	Table 3
Function $Q(s)$	Equation (62)		Equation (52)	$\bar{\Omega}^2(\omega, \bar{P}, C)\tilde{n}^2(s)$ and equations (66) and (67)	
Purpose	Asymptotic behavior of second order governing equation		Both governing equations with same qualitative behavior	Second order equation as an exact governing equation	

and $g(s)$, describing the height non-uniformity, are

$$m(s) = \begin{cases} 1 & \text{for } s \leqslant 0 \text{ and } s \geqslant \bar{L}, \\ 1 + \dfrac{A}{h_0}(f(s) - g(s)) & \text{for } 0 < s < \bar{L}, \end{cases} \tag{72}$$

$$ei(s) = \begin{cases} 1 & \text{for } s \leqslant 0 \text{ and } s \geqslant \bar{L}, \\ \left(1 + \dfrac{A}{h}(f(s) - g(s))\right)^3 + 3\left(1 + \dfrac{A}{h}(f(s) - g(s))\right)\left(\dfrac{A}{h}(f(s) + g(s))\right)^2 & \text{for } 0 < s < \bar{L}. \end{cases} \tag{73}$$

A is the amplitude of the height non-uniformity.

5.1.1. *Constant tensile force*

For the first two examples the tensile force is assumed constant. If we substitute equations (72) and (73) into equation (62) for the coefficient function $Q(s)$, we obtain its expression in terms of the functions $f(s)$ and $g(s)$.

The function $f(s)$ for the first two examples is

$$f(s) = \cos(k_g \lambda s + \gamma). \tag{74}$$

The parameter $k_g \lambda$ is the non-dimensional wavenumber of the non-uniformity and here γ is a phase factor. We consider that $k_g = 2\pi$, which implies a non-uniformity with wavelength equal to 1 m. The size of the non-uniformity is equal to 22 times its wavelength. The function $g(s)$ is specified in terms of the function $f(s)$ according to Table 1.

For the first example, the non-uniformity comes from the flexural rigidity, and in its expression we have the square of the parameter A/h_0. Therefore, the actual non-uniformity magnitude is $(A/h_0)^2$. The actual wavelength of the non-uniformity is equal to half the wavelength of the function $f(s)$.

For the second example, the non-dimensional mass per unit length is a linear function of $f(s)$, and the magnitude of the uniformity is given by the parameter A/h_0. For the non-dimensional flexural rigidity, the magnitude in the non-uniformity is now given by the cube of the parameter A/h_0. The non-dimensional mass per unit length and the non-dimensional flexural rigidity have the same periodicity as the function f(s). Values of the non-uniformity parameter A/h_0 are given in Table 1.

Results regarding the first and second examples are illustrated, respectively, in Figures 1 and 2. A detailed description of Figures 1 and 2 is given in the second and third column of Table 2. According to parts (b) and (d) of Figures 1 and 2, the agreement between the results for the modulus of the reflection coefficient from the numerical simulation of both governing equations increases as the magnitude (parameter A/h_0) of the geometric non-uniformity decreases. This illustrates the asymptotic nature of the second order governing equation with respect to the steepness in the beam non-uniformity.

The peak in the reflection coefficient in part (a) of Figure 1 and in parts (a) and (c) of Figure 2 is due to the Bragg scattering phenomenon. Bragg scattering peaks in the reflection coefficient occur when the ratio between the wavenumber of the incident wave multiplied by

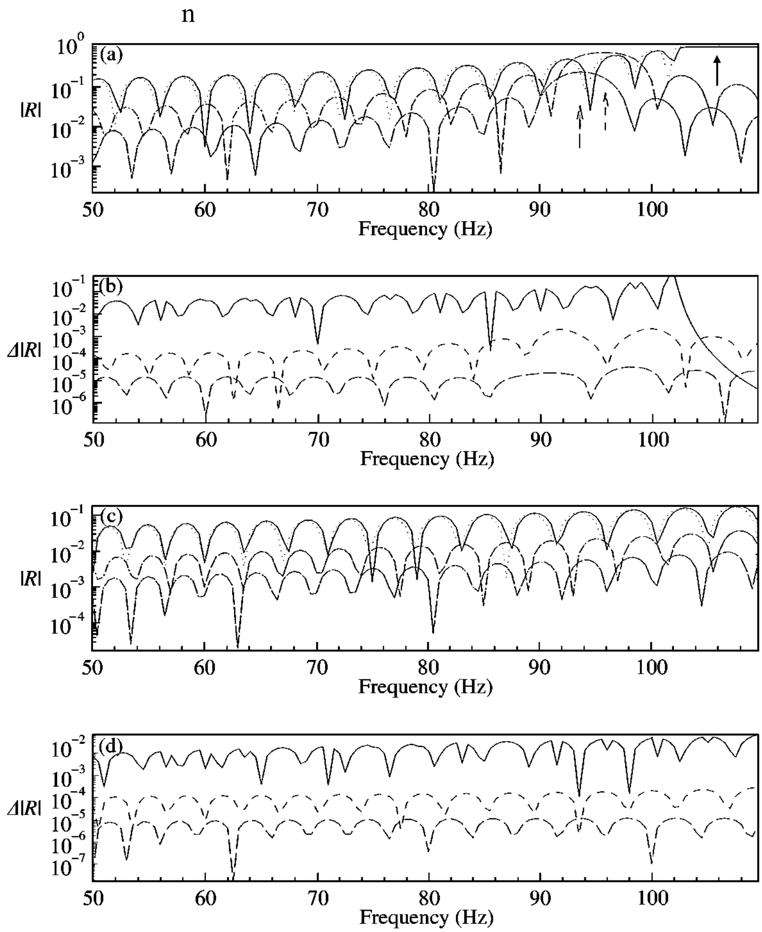

Figure 1. Reflection coefficient as a function of wave frequency of the incident wave. $|R|$: modulus of the reflection coefficient. $\Delta|R|$: difference between modulus of the reflection coefficient. Vertical arrows: Bragg scattering peaks. In parts (a) and (c), lines ———, ------- and – – – –: numerical simulation of the full governing equation (6), respectively, for $A/h_0 = 0{\cdot}2$, 0·1 and 0·05; lines ········, -·-·-·-·-· and -·-·-·-·-·-: numerical simulation of the second order governing equation (50) for $A/h_0 = 0{\cdot}2$, 0·1 and 0·05. In parts (b) and (d), lines ———, ------- and – – – – –: difference between modulus of the reflection coefficient, respectively, for $A/h_0 = 0{\cdot}2$, 0·1 and 0·05.

2 and the non-uniformity wavenumber is a natural number. The Bragg scattering peak (arrow) illustrated in Figures 1 and 2 are first Bragg scattering peaks.

5.1.2. *Variable tensile force*

Results in Figure 3 for the third example illustrate the fact that the qualitative behavior predicted by the second order governing equation is in agreement with the results obtained using the full governing equation. Results presented in Figure 3 are described in the fourth column of Table 2.

For the third example, the function $f(s)$ is

$$f(s) = -\operatorname{sech}(k_g\lambda(s - s_0)). \tag{75}$$

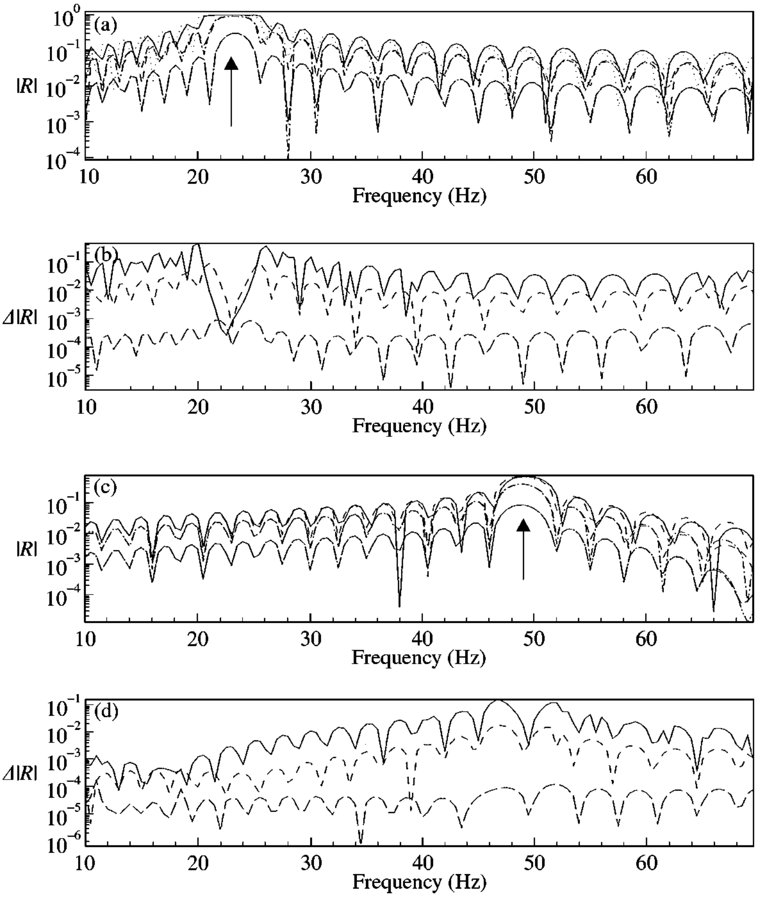

Figure 2. Reflection coefficients as a function of wave frequency of the incident wave. $|R|$: modulus of the reflection coefficient. $\Delta|R|$: difference between modulus of the reflection coefficient. Vertical arrows: Bragg scattering peaks. In parts (a) and (c), lines ———, ------- and ----: numerical simulation of the full governing equation (6), respectively, for $A/h_0 = 0{\cdot}1$, $0{\cdot}05$ and $0{\cdot}01$; lines ········, -·-·-·-·-· and -·-·-·-·-·: numerical simulation of the second order governing equation (50), respectively, for $A/h_0 = 0{\cdot}1$, $0{\cdot}05$ and $0{\cdot}01$. In parts (b) and (d), lines ———, ------- and -----: difference between modulus of the reflection coefficient, respectively, for $A/h_0 = 0{\cdot}1$, $0{\cdot}05$ and $0{\cdot}01$.

The product $A\,k_g\lambda/h_0$ gives the magnitude of this non-uniformity steepness. The function $g(s)$ is related to the function $f(s)$ according to Table 1. The non-dimensional tensile force is given as

$$\bar{P}(s) = -\,A_p\,\mathrm{sech}\,(k_p\lambda(s - s_0)), \tag{76}$$

where A_p gives the magnitude of the non-dimensional non-uniformity in the tensile load, and the product $A_p k_p \lambda$ gives its steepness.

The non-dimensional mass per unit length and the non-dimensional flexural rigidity reaches a maximum or minimum at $s = s_0$, and the non-dimensional tensile force reaches a maximum negative value at $s = s_0$. We may choose the values of the non-uniformity parameters A/h_0, A_p, k_g and k_p such that the coefficient function $Q(s)$ reaches negative values for intervals of the space co-ordinate. When this happens, the second order equation (50) predicts exponentially small transmission and almost complete reflection.

For part (b) of Figure 3, we chose the parameters A/h_0, A_p, k_g and k_p such that the coefficient function $Q(s)$ assumes negative values for two intervals of the space co-ordinate.

TABLE 2

Description of the figures

Figure	1	2	3	4	5
Results	Modulus of the reflection coefficient for left wave incidence as a function of the wave frequency				
Part (a) description	Modulus of the reflection coefficient from the numerical simulation of equations (6) and (50) for 3 values of A/h_0 and with $P = 0\text{N}$		Modulus of the reflection coefficient from the numerical simulation of equations (6) and (50) with $A/h_0 = 0{\cdot}05$ and $0{\cdot}01$, $A_P = 0\text{N}$, $k_P = 0$ and $k_g = 2$	Modulus of the reflection coefficient from the numerical simulation of equations (6) and (50) with $A = 0{\cdot}2$, $0{\cdot}1$ and $0{\cdot}05$ and $P = 0\text{N}$	Modulus of the reflection coefficient from the numerical simulation of equation (6) and from equation (79) with $k_g = 10$ and 5
Part (b) description	Difference between the modulus of the reflection coefficient from the numerical simulation of equations (6) and (50) for three values of A/h_0 with $P = 0\,\text{N}$		Same as part (a), but with $A_P = 10\,000\,\text{N}$, $k_p = 10$ and $k_g = 2$	Same as part (a), but with $P = 10\,000\,\text{N}$	Same as part (a)
Part (c) description	Same as part (a), but with $P = 10\,000\,\text{N}$		Intervals of s where $Q(s) < 0$ as a function of wave frequency		Same as part (a)
Part (d) description	Same as part (b), but with $P = 10\,000\,\text{N}$				Same as part (a)
Part (e) description					Non-dimensional flexural rigidity distributions
Resonance	First Bragg scattering peak in part (a)	First Bragg scattering peak in parts (a) and (c)		First and second Bragg scattering peaks in part (a) and first peak in part (b)	

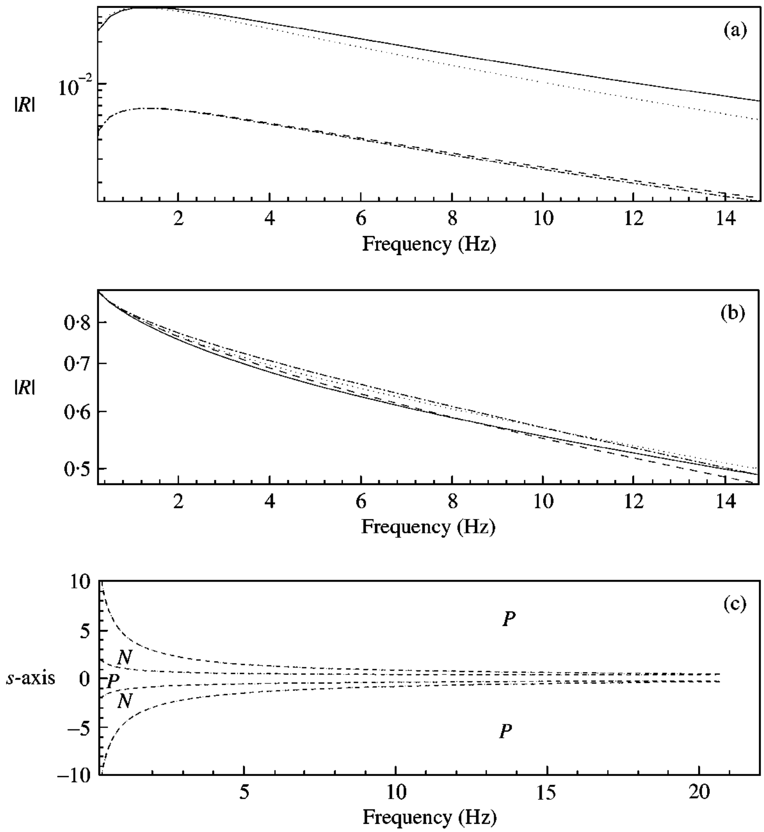

Figure 3. Modulus of the reflection coefficient as a function of wave frequency of the incident wave. In parts (a) and (b), lines ——— and - - - - - - -: numerical simulation of the full governing equation (6), with A/h_0, respectively, 0·05 and 0·01; lines ········· and - · - · - · - · -: numerical simulation of the second order governing equation (50) with A/h_0, respectively, 0·05 and 0·01. Part (c) displays contour plots for $Q(s) = 0$. The letters P and N in part (c) stand for positive and negative values of $Q(s)$. Line - - - - - - -: $A_P = 10\,000$ N, $k_p = 10$, $k_g = 2$ and $A/h_0 = 0{\cdot}05$, and line - · - · - · - · -: $A_P = 10\,000$ N, $k_p = 10$, $k_g = 2$ and $A/h_0 = 0{\cdot}01$.

In this case, large wave reflection is observed. The size of the intervals where $Q(s)$ is negative is illustrated in part (c) of Figure 3 as a function of the frequency of the incident wave.

5.2. TENSILE FORCE AND THE MASS PER UNIT LENGTH TIMES FLEXURAL RIGIDITY ARE CONSTANT

For the fourth and fifth examples, the tensile force and the product of the flexural rigidity by the mass per unit length are constants. Under such conditions, the results predicted by both governing equations are the same, as illustrated in Figures 4 and 5. These figures are described, respectively, in the fifth and sixth columns of Table 2.

The fourth example is described by the fifth column of Table 1. The non-dimensional flexural rigidity for this example is prescribed according to the equation

$$ei(s) = 1 + \frac{A}{2}\cos(k_g\lambda s + \gamma)(\tanh(\kappa(s + s_0)) - \tanh(\kappa(s - s_0))), \tag{77}$$

which is a periodic function along a finite part of the beam and otherwise zero.

The fifth example is described in the sixth column of Table 1. The non-dimensional flexural rigidity for this example is

$$ei(s) = S - N\frac{\exp(k_g\lambda(s+s_0))}{1+\exp(k_g\lambda(s+s_0))} - 4M\frac{\exp(k_g\lambda(s+s_0))}{(1+\exp(k_g\lambda(s+s_0)))^2}. \quad (78)$$

For $N = 0$, equation (78) is basically the square of the hyperbolic secant function, and for $M = 0$, equation (78) gives basically the hyperbolic tangent function. The second order equation for this non-uniformity has an analytic solution in terms of hyper-geometric functions, as discussed in Appendix D. The reflection and transmission coefficients are given in terms of gamma functions by the equations

$$R^- = \frac{\Gamma(\beta)\Gamma(1-\gamma)\Gamma(\gamma-\delta)}{\Gamma(\beta-\gamma+1)\Gamma(\gamma-1)\Gamma(1-\delta)}, \quad (79)$$

$$T^- = \frac{\Gamma(\beta)\Gamma(\gamma-\delta)}{\Gamma(\beta-\delta+1)\Gamma(\gamma-1)}. \quad (80)$$

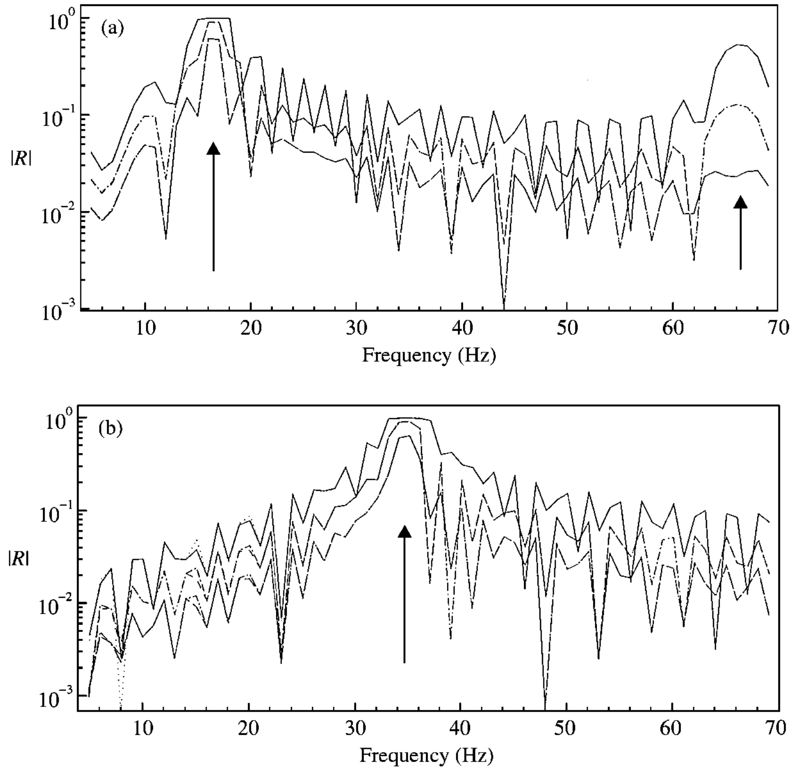

Figure 4. Modulus of the reflection coefficient as a function of wave frequency of the incident wave. Non-dimensional flexural rigidity and mass per unit length defined by equations (77) and (C.3), with $C = 1$, $\kappa = 20$, $k_g = 2\pi$, $s_0 = 10{\cdot}25$ and $\gamma = 0$. Vertical arrow: Bragg scattering peak. Lines ——, ------ and – – – – –: numerical simulation of the full governing equation (6) with A, respectively, equal to 0·2, 0·1 and 0·05. Lines ········, -·-·-·-·- and -·-·-·-·-: numerical simulation of the second order governing equation (50) with A, respectively, equal to 0·2, 0·1 and 0·05.

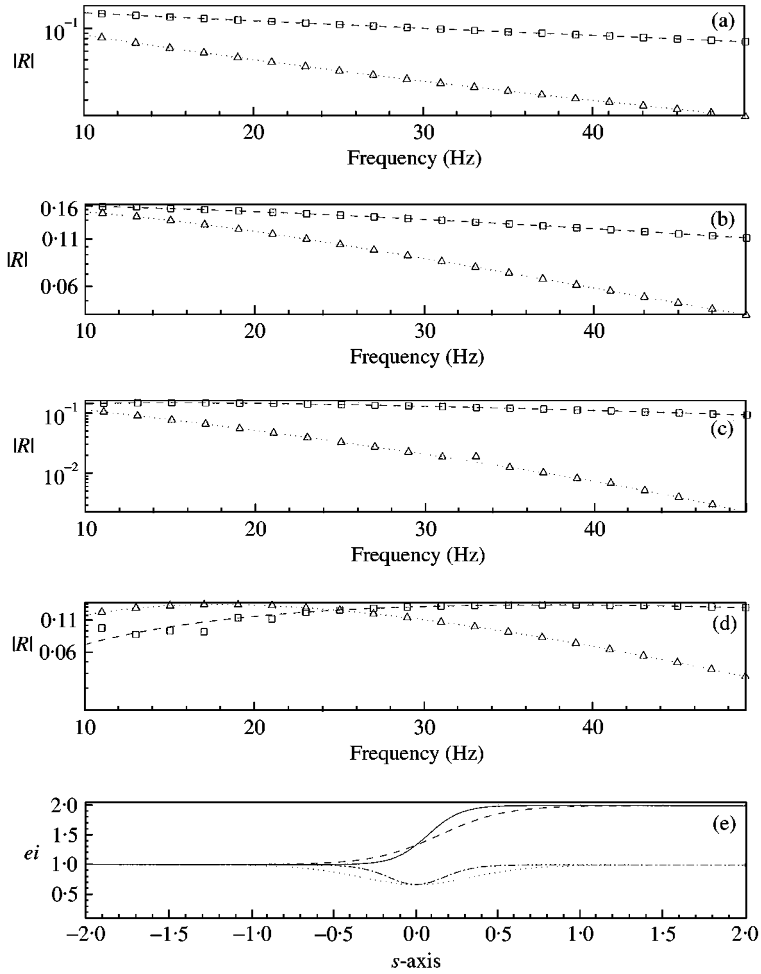

Figure 5. Modulus of the reflection coefficient as a function of wave frequency of the incident wave. Non-dimensional flexural rigidity and mass per unit length defined by equations (78) and (C.3) with $C = 1$. Lines ----- and ········: numerical simulation of the full governing equation (6). Symbols □ and △: closed form solution given by equations (78) and (81). Line ------- and symbol □: $k_g = 10$. Line ······ and symbol △: $k_g = 5$. Values of the parameters S, N, M and s_0 and the tensile force P for parts (a), (b), (c) and (d) are listed in Table 3. In part (e), lines ——— and ------: non-dimensional flexural rigidity distribution used in parts (a) and (b) with k_g, respectively, equal to 10 and 5, and lines -·-·-·-·-· and ········: non-dimensional flexural rigidity distribution used in parts (c) and (d) with k_g, respectively, equal to 10 and 5.

The quantities β, γ and δ are given in terms of the parameters $\tilde{\Omega}(\omega, \bar{P}, C)$, k_g, λ, S, N and M according to the equations

$$\gamma = 1 + \mathrm{i}2\,\frac{\tilde{\Omega}(\omega, \bar{P}, C)}{|k_g\lambda|}\sqrt{S}, \tag{81}$$

$$\delta = \frac{1}{2} + \frac{1}{2}\left[1 - \frac{16\tilde{\Omega}(\omega, \bar{P}, C)^2 M}{(k_g\lambda)^2}\right] + \mathrm{i}\,\frac{\tilde{\Omega}(\omega, \bar{P}, C)}{|k_g\lambda|}(\sqrt{S} - \sqrt{S - N}), \tag{82}$$

$$\beta = \frac{1}{2} + \frac{1}{2}\left[1 - \frac{16\tilde{\Omega}(\omega, \bar{P}, C)^2 M}{(k_g\lambda)^2}\right] + \mathrm{i}\,\frac{\tilde{\Omega}(\omega, \bar{P}, C)}{|k_g\lambda|}(\sqrt{S} + \sqrt{S - N}), \tag{83}$$

TABLE 3

Parameters for the non-dimensional flexural rigidity and tensile force values

Figure	Parameter S	Parameter N	Parameter M	Parameter s_0	Tensile force P
5(a)	1·0	0·5	0·0	0·0	0 N
5(b)	1·0	0·5	0·0	0·0	10 000 N
5(c)	1·0	0·0	− 0·5	0·0	0 N
5(d)	1·0	0·0	− 0·5	0·0	10 000 N

as suggested by Brekhovskikh [11, p. 55]. The parameter $\tilde{\Omega}(\omega, \bar{P}, C)$ is defined by equation (67).

According to Figure 5, the modulus of the reflection coefficient from the numerical simulation of the full governing equation (6) agrees with the modulus of the reflection coefficient given by equation (79). The non-uniformity parameters used in parts (a)–(d) of Figure 5 are displayed in Table 3. Part (e) of Figure 5 illustrates the distributions of the non-dimensional flexural rigidity along the beam used to generate the results in parts (a)–(d) of Figure 5.

6. APPLICATIONS—THE DESIGN PROBLEM

We are going to design the non-uniformity in the mass per unit length and in the flexural rigidity of a Bernoulli–Euler beam such that the relationship between the reflection coefficient and the wavelength of the incident wave has a prescribed form.

We assume the tensile force applied along the beam and the product of the mass per unit length by the flexural rigidity of the beam to be constants. Under these restrictions, the second order governing equation is an exact governing equation for wave propagation along the non-uniform Bernoulli–Euler beam. We consider a non-uniformity shape such that the second order governing equation for the Bernoulli–Euler beam has a known analytical solution. In this situation, the relation between the reflection coefficient and the wavelength of the incident wave can be derived in closed form. The parameters describing the non-uniformity shape, i.e., its steepness, size and amplitude, are also the parameters of the relation between the reflection coefficient and the wavelength of the incident wave. Therefore, if we prescribe the form of the relationship between the reflection coefficient and the wavelength of the incident wave, we can determine the desired shape of the non-uniformity in mass per unit length and in flexural rigidity.

We consider a Bernoulli–Euler beam consisting of three uniform pieces with a constant tensile force P applied to it. The distribution of mass per unit length and flexural rigidity for each piece is defined by the equations

$$\rho A(x) = \begin{cases} \rho A_0 & \text{for } x < 0 \text{ and } x > L, \\ \rho A_1 & \text{for } 0 \leqslant x \leqslant L, \end{cases} \tag{84}$$

$$EI(x) = \begin{cases} EI_0 & \text{for } x < 0 \text{ and } x > L, \\ EI_1 & \text{for } 0 \leqslant x \leqslant L. \end{cases} \tag{85}$$

The restriction that the product of the mass per unit length by the flexural rigidity is a constant is expressed by the equation

$$\rho A_0 EI_0 = \rho A_1 EI_1 = C''. \tag{86}$$

The value of the constant C'' is fixed by equation (86), since the quantities ρA_0 and EI_0 are assumed known.

The second order governing equation for the Euler–Bernoulli beam under such restrictions assumes the form (69), and its coefficient function in terms of the beam parameters is

$$\tilde{\Omega}^2(\Omega, P, C'')n(x)^2 = \begin{cases} k_0^2 = \dfrac{1}{EI_0}\left\{-\dfrac{P}{2} + \dfrac{1}{2}\sqrt{P^2 + 4\Omega^2 C''}\right\} & \text{for } x < 0 \text{ and } x > L, \\ k_1^2 = \dfrac{1}{E_1 I_1}\left\{-\dfrac{P}{2} + \dfrac{1}{2}\sqrt{P^2 + 4\Omega^2 C''}\right\} & \text{for } 0 \leqslant x \leqslant L. \end{cases} \tag{87}$$

The quantity Ω is the dimensional frequency of the incident wave. k_0 and k_1 are wavenumbers.

According to equation (87), the wavenumber k_1 can be written in terms of the wavelength λ of the incident wave, as follows:

$$k_1 = \frac{EI_0}{EI_1}\frac{2\pi}{\lambda}. \tag{88}$$

Equation (69) has an analytical solution for the function $\tilde{\Omega}^2(\Omega, P, C'')n(x)^2$ defined by equation (87). We consider the radiation condition of left wave incidence given by equation (56) with $k_1^{\pm} = k_0$.

The reflection coefficient for left wave incidence can be written in terms of the wavenumbers k_0 and k_1 and in terms of the size L of the part of the beam with unknown properties, according to the equation

$$\tilde{R}^- = \frac{\alpha(1-\beta)}{1+\alpha^2\beta}, \tag{89}$$

where the quantities, α, β and Γ are defined according to the equations

$$\alpha = \frac{k_0^2 - k_1^2}{k_0 + k_1} = \frac{EI_1 - EI_0}{EI_1 + EI_0}, \tag{90}$$

$$\Gamma = 4k_1 L = \frac{8\pi}{\lambda}\frac{EI_0}{EI_1}L, \qquad \beta = \exp(\mathrm{i}\Gamma), \tag{91, 92}$$

The modulus of the reflection coefficient $\tilde{R}^-$ is

$$|\tilde{R}^-| = \left\{\left(\frac{\alpha(1-\cos(\Gamma))(1-\alpha^2\cos(\Gamma))}{(1-\alpha^2\cos(\Gamma))^2 + \alpha^4(\sin(\Gamma))^2} + \frac{\alpha^3(\sin(\Gamma))^2}{(1-\alpha^2\cos(\Gamma))^2 + \alpha^4(\sin(\Gamma))^2}\right)^2 + \left(-\frac{\alpha\sin(\Gamma)(1+\alpha^2\cos(\Gamma))}{(1-\alpha^2\cos(\Gamma))^2 + \alpha^4(\sin(\Gamma))^2} + \frac{\alpha^3(1-\cos(\Gamma))\sin(\Gamma)}{(1-\alpha^2\cos(\Gamma))^2 + \alpha^4(\sin(\Gamma))^2}\right)^2\right\}^{1/2}. \tag{93}$$

The reflection coefficient $\tilde{R}^-$ is an oscillatory function of the wavelength λ of the incident wave, according to equations (91) and (93). According to equation (93), the reflection coefficient attains zero value at a discrete set of wavelengths of the incident wave. These are

TABLE 4

Design problem steps

Step	Description
Constant C''	Given by equation (86) in terms of the known quantities ρA_0 and EI_0
Chose λ_a	We chose a desired wavelength of perfect transmission. No restriction posed on this choice
Chose λ_b	Next wavelength of perfect transmission. It has to satisfy the condition of perfect transmission (94), which can be written as $\lambda_b = (n/(n-1))\lambda_a$ for n a natural number. This relation fixes the value of n
Chose $\lvert R_{max}\rvert$	We chose the maximum value of the modulus of reflection coefficient for $\lambda_a < \lambda < \lambda_b$. According to equations (92) and (93), $(\mathrm{d}/\mathrm{d}\lambda)\lvert R\rvert = 0 \rightarrow (\mathrm{d}/\mathrm{d}\Gamma)\lvert R\rvert = 0$ implies $\Gamma = m\pi$ with m an integer number. $\Gamma = \pi + 2m\pi$, with m integer, gives $\lvert R_{max}\rvert = 2\alpha/(\alpha^2+1)$
Obtain α	This quantity is defined by equation (90). In terms of $\lvert R_{max}\rvert$, we chose $\alpha = \frac{1}{\lvert R_{max}\rvert}(1-\sqrt{1-\lvert R_{max}\rvert^2})$
Obtain EI_1	Given in terms of α as $EI_1 = ((\alpha+1)/(1-\alpha))\,EI_0$
Obtain L	The size of the part of the beam to be designed, denoted as L, is given by the equation $L = n\lambda_a EI_1/4EI_0 = (n\lambda_a/4)\,(\alpha+1)/(1-\alpha)$
Obtain ρA_1	The mass per unit length of the part to be designed is given by the equation $\rho A_1 = C''/EI_1$

the wavelengths for perfect wave transmission. They satisfy the condition for perfect transmission,

$$1 - \exp(\mathrm{i}\Gamma) = 0. \tag{94}$$

We design the quantities ρA_1, EI_1 and L to have perfect transmission at chosen wavelengths λ_a and λ_b ($\lambda_b > \lambda_a$). We also prescribe the maximum value that the modulus of the reflection coefficient assumes for wavelengths between λ_a and λ_b. Once we prescribe a wavelength of perfect transmission, i.e., λ_a, the next wavelength of perfect transmission, i.e., λ_b, could not be prescribed in an arbitrary way.

First, we chose two desired wavelengths of perfect transmission, i.e., λ_a and λ_b, but with λ_b as a function of λ_a such that the condition for perfect transmission, as given in the third line of Table 4, is satisfied. Second, we chose the maximum value of the modulus of the reflection coefficient for $\lambda_a < \lambda < \lambda_b$. With these three quantities chosen, we follow the steps given in Table 4.

In Figure 6, we present plots of equation (93) as a function of the wavelength of the incident wave, and we compare them with the values of the reflection coefficient obtained from the exact solution of the full governing equation. These values of the modulus of the reflection coefficient have been obtained in Appendix E.

We also present Table 5 with the values of the designed mass per unit length ρA_1 and flexural rigidity EI_1, respectively, as a fraction of the known values of the mass per unit length ρA_0 and flexural rigidity EI_0. We also give the chosen wavelengths of perfect transmission λ_a and λ_b, the size L and the maximum value of the modulus of the reflection coefficient $\lvert\tilde{R}^-\rvert_{max}$. These values presented in Table 5 are associated with Figures 6(a)–(d).

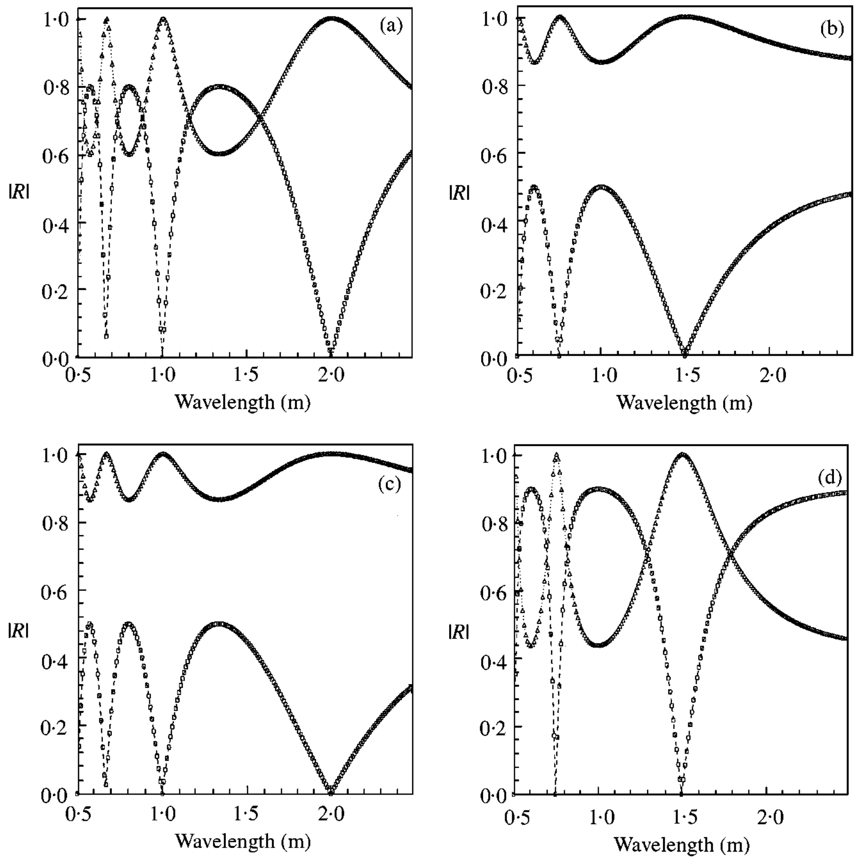

Figure 6. Modulus of the reflection and transmission coefficients as a function of wavelength of the incident wave for the designed Euler–Bernoulli beam. $|\tilde{R}^-|$ from equation (93): ------. $|\tilde{T}^-| = \sqrt{1-|\tilde{R}^-|^2}$: ········. Numerical simulation for the reflection coefficient (E.13): □. Numerical results for the transmission coefficient (E.13): △. Properties of the designed beams for (a), (b), (c) and (d) are given in Table (5).

TABLE 5

Parameters of the designed Bernoulli–Euler beam

Figure	Ratio $\rho A_1/\rho A_0$	Ratio EI_1/EI_0	λ_a	λ_b	$\lvert R^-\rvert_{max}$	Length L
6(a)	0·11	9·0	1·0 m	2·0 m	0·8	2·736 m
6(b)	0·33	3·0	1·0 m	2·0 m	0·5	1·914 m
6(c)	0·33	3·0	0·75 m	1·5 m	0·5	1·436 m
6(d)	0·0526315789	19·0	0·75 m	1·5 m	0·9	2·747 m

7. DISCUSSION AND CONCLUSIONS

In the previous section, we illustrated how to use a Helmholtz-like second order differential equation with a known solution to design a non-uniform Bernoulli–Euler beam with constants tensile force and product of the flexural rigidity by the mass per unit length such that the functional relation between the reflection coefficient and the wavelength of the incident wave had a prescribed form. This approach to the design problem is limited, since there are few Helmholtz-like second order differential equations with variable coefficient that have analytical solutions. To handle general design problems, a more general approach is necessary.

We discuss how to apply inverse-scattering techniques for the one-dimensional Helmholtz equation in the half line to the design problem of a Bernoulli–Euler beam under constant tensile force and with the product of the mass per unit length by the flexural rigidity being a constant.

Under the restriction mentioned above, wave propagation along a non-uniform Bernoulli–Euler beam is governed by the second order differential equation (69). The relationship between the frequency of the incident wave Ω and the "wave frequency" $\tilde{\Omega}$ is given by equation (71). The constant tensile force P enters as a parameter in equation (71) and the constant C' is the value of the product of the mass per unit length times the flexural rigidity along the entire beam.

Once the value of the tensile force is prescribed, the relationship between Ω and $\tilde{\Omega}$ is one-to-one for $\Omega \geqslant 0$. Then, the knowledge of the functional relation between the reflection coefficient and frequency of the incident wave Ω implies that we know the functional relation between the reflection coefficient and the "wave frequency" $\tilde{\Omega}$. Therefore, we can apply inverse-scattering techniques developed for the Helmholtz equation on the half line to design the index of refraction $n(x)$ of equation (69) for $x > 0$, given a prescribed functional relation between the reflection coefficient at $x = 0$ and the "wave frequency" $\tilde{\Omega}$. Designing the index of refraction of equation (69) implies designing the inverse of the flexural rigidity of the Bernoulli–Euler beam. Once the flexural rigidity is obtained from the design of the index of refraction $n(x)$, the mass per unit length follows from the restriction given by equation (70).

Regarding techniques to solve the inverse problem for the Helmholtz equation on the half line, we can mention the non-linear approximate method described in Jaggard and Kim [12] and the exact method based on layer-stripping technique described in references [9, 10]. These techniques to handle the inverse–scattering problem can be applied to design the non-uniformity in a Bernoulli–Euler beam under the restrictions of the tensile force and the product of the mass per unit length by the flexural rigidity being constants, as discussed above. Bernoulli–Euler beams satisfying the restrictions described above can be built, and they could be useful in engineering applications, where passive vibration isolation is desired.

According to the numerical results displaced in section 5, the second order governing equation (50) recovers the behavior predicted by the full governing equation (6) for general non-uniformities when its steepness is small. The second order governing equation (50) is useful in studying the wave-scattering phenomenon along weakly non-uniform Bernoulli–Euler beams of many wavelengths. Under the restriction of the tensile force and the product of the mass per unit length by the flexural rigidity being constants, the second order governing equation (50) is able to predict wave propagation along the Bernoulli–Euler beam even for large deviation from uniformity, since it is an exact governing equation.

Higher order WKB or phase integral methods can be applied to the new governing equation to obtain analytic approximations for the beam scattering quantities, since for second order equations these methods are nowadays well developed. For an account on the WKB method and phase integral methods see reference [13].

REFERENCES

1. M. A. HAWWA 1997 *Journal of Sound and Vibration* **199**, 453–461. Reflection of flexural waves in geometrically periodic beams.

2. C. PIERRE 1988 *Journal of Sound and Vibration* **126**, 485–512. Mode localization and eigenvalue loci veering phenomena in disorderd structures.
3. M. S. TRIANTAFYLLOU and G. S. TRIANTAFYLLOU 1991 *Journal of Sound and Vibration* **150**, 485–500. Frequency coalescence and mode localization phenomena: a geometric theory.
4. S. Y. LEE and H. Y. KE 1992 *Journal of Applied Mechanics* **59**, 189–196. Flexural wave propagation in an elastic beam with periodic structure.
5. S. NAGULESWARAN 1995 *Journal of Sound and Vibration* **187**, 311–327. The vibration of a "complete" Euler–Bernoulli beam of constant depth and breadth proportional to axial co-ordinate raised to a positive exponent *s*.
6. S. ABRATE 1995 *Journal of Sound and Vibration* **185**, 703–716. Vibration of non-uniform rods and beams.
7. J. HEADING 1977 *Mathematical Proceedings of Cambridge Philosophical Society* **81**, 121–132. Deductions from some artificial refractive index profiles based on the elementary functions.
8. A. A. MERAB 1987 *Ph.D. Thesis, Joint Program in Oceanography and Ocean Engineering, Woods Hole Oceanographic Institution and Massachusetts Institute of Technology*. Exact reconstruction of ocean bottom velocity profiles from monochromatic scattering data.
9. J. SYLVESTER, D. WINEBRENNER and F. GYLS-COLWELL 1996 *SIAM Journal of Applied Mathematics* **56**, 736–754. Layer stripping for the Helmholtz equation.
10. J. SYLVESTER and D. WINEBRENNER 1998 *SIAM Journal of Applied Mathematics* **59**, 669–699. Linear and nonlinear inverse scattering.
11. L. M. BREKHOVSKIKH and O. A. GODIN 1992 *Acoustics of Layered Media I. Plane Waves and Quasi-Plane Waves*. Springer Series on Wave Phenomena, New York: Springer Verlag.
12. D. L. JAGGARD and Y. KIM 1985 *Journal of the Optical Society of America* **2**, 1922–1930. Accurate one-dimensional inverse scattering using a nonlinear renormalization technique.
13. N. FRÖMAN and P. O. FRÖMAN 1995 *Phase-Integral Method Allowing Nearlying Transition Points*. Springer Tracts in Natural Philosophy, New York: Springer Verlag.

APPENDIX A: GOVERNING EQUATION FOR QUANTITIES $\tilde{A}(S)$, $\tilde{B}(S)$, $\tilde{C}(S)$ AND $\tilde{D}(S)$, AND ELEMENTS OF THE MATRIX $M(S)$.

Here, we present the governing equations for the dependent variables $\tilde{A}(s)$, $\tilde{B}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$ derived in section 4.1. This set of equations follows.

$$\frac{\mathrm{d}\tilde{A}}{\mathrm{d}s} + \frac{\mathrm{d}\tilde{B}}{\mathrm{d}s} + \frac{\mathrm{d}\tilde{C}}{\mathrm{d}s} + \frac{\mathrm{d}\tilde{D}}{\mathrm{d}s} - \mathrm{i}\,k_1(s)\tilde{A}(s) + \mathrm{i}\,k_1(s)\tilde{B}(s) = 0, \tag{A.1}$$

$$- k_s(s)\tilde{C}(s) + k_2(s)\tilde{D}(s) = 0, \tag{A.2}$$

$$\begin{aligned} &\mathrm{i}k_1(s)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}s} - \mathrm{i}k_1(s)\frac{\mathrm{d}\tilde{B}}{\mathrm{d}s} + k_2(s)\frac{\mathrm{d}\tilde{C}}{\mathrm{d}s} - k_2(s)\frac{\mathrm{d}\tilde{D}}{\mathrm{d}s} \\ &+ \left(\mathrm{i}\frac{\mathrm{d}k_1}{\mathrm{d}s} + k_1^2(s)\right)\tilde{A}(s) + \left(-\mathrm{i}\frac{\mathrm{d}k_1}{\mathrm{d}s} + k_1^2(s)\right)\tilde{B}(s) \\ &+ \left(\frac{\mathrm{d}k_2}{\mathrm{d}s} - k_2^2(s)\right)\tilde{C}(s) - \left(\frac{\mathrm{d}k_2}{\mathrm{d}s} - k_2^2(s)\right)\tilde{D}(s) = 0, \end{aligned} \tag{A.3}$$

$$\begin{aligned} &\Bigg([-k_1^2(s) - \mathrm{i}k_1(s)(k_2^2(s) - k_1^2(s))]\frac{\mathrm{d}\tilde{A}}{\mathrm{d}s} + [-k_1^2(s) + \mathrm{i}k_1(s)(k_2^2(s) - k_1^2(s))]\frac{\mathrm{d}\tilde{B}}{\mathrm{d}s} \\ &+ [k_2^2(s) - k_2(s)(k_2^2(s) - k_1^2(s))]\frac{\mathrm{d}\tilde{C}}{\mathrm{d}s} + [k_2^2(s) + k_2(s)(k_2^2(s) - k_1^2(s))]\frac{\mathrm{d}\tilde{D}}{\mathrm{d}s}\Bigg) ei(s) \end{aligned}$$

$$-\left(ei(s)\left[2\,k_1(s)\frac{\mathrm{d}k_1}{\mathrm{d}s}+\mathrm{i}(k_2^2(s)-k_1^2(s))\frac{\mathrm{d}k_1}{\mathrm{d}s}-\mathrm{i}\,k_1^3(s)\right]+k_1^2(s)\frac{\mathrm{d}ei}{\mathrm{d}s}\right)\tilde{A}(s)$$

$$-\left(ei(s)\left[2\,k_1(s)\frac{\mathrm{d}k_1}{\mathrm{d}s}-\mathrm{i}(k_2^2(s)-k_1^2(s))\frac{\mathrm{d}k_1}{\mathrm{d}s}+\mathrm{i}\,k_1^3(s)\right]+k_1^2(s)\frac{\mathrm{d}ei}{\mathrm{d}s}\right)\tilde{B}(s)$$

$$+\left(ei(s)\left[2\,k_2(s)\frac{\mathrm{d}k_2}{\mathrm{d}s}-(k_2^2(s)-k_1^2(s))\frac{\mathrm{d}k_2}{\mathrm{d}s}-k_2^3(s)\right]+k_2^2(s)\frac{\mathrm{d}ei}{\mathrm{d}s}\right)\tilde{C}(s)$$

$$+\left(ei(s)\left[2\,k_2(s)\frac{\mathrm{d}k_2}{\mathrm{d}s}+(k_2^2(s)-k_1^2(s))\frac{\mathrm{d}k_2}{\mathrm{d}s}+k_2^3(s)\right]+k_2^2(s)\frac{\mathrm{d}ei}{\mathrm{d}s}\right)\tilde{D}(s)=0, \qquad \text{(A.4)}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}(-\mathrm{i}k_1(s)(k_1^2(s)+ei(s)[k_2^2(s)-k_1^2(s)]))(\tilde{A}(s)-\tilde{B}(s))\\&+\frac{\mathrm{d}}{\mathrm{d}s}(k_2(s)(k_2^2(s)-ei(s)[k_2^2(s)-k_1^2(s)]))(\tilde{C}(s)-\tilde{D}(s))\\&+(-\mathrm{i}k_1(s)(k_1^2(s)+ei(s)[k_2^2(s)-k_1^2(s)]))\left(\frac{\mathrm{d}\tilde{A}}{\mathrm{d}s}-\frac{\mathrm{d}\tilde{B}}{\mathrm{d}s}\right)\\&+(k_2(s)(k_2^2(s)-ei(s)[k_2^2(s)-k_1^2(s)]))\left(\frac{\mathrm{d}\tilde{C}}{\mathrm{d}s}-\frac{\mathrm{d}\tilde{D}}{\mathrm{d}s}\right)=0.\end{aligned} \qquad \text{(A.5)}$$

These sets of equations can be written in matrix form, according to equation (15). The elements of the matrix, denoted as $\mathbf{M}(s)$, of this system of equations are given in terms of the non-dimensional flexural rigidity $ei(s)$, non-dimensional tensile force $\bar{P}(s)$ and in terms of the wavenumbers $k_1(s)$ and $k_2(s)$, which are given, respectively, by equations (10) and (11). The equations for the elements of matrix $\mathbf{M}(s)$ follows:

$$M_{11}(s)=-\frac{1}{2k_1(s)}\frac{\mathrm{d}k_1}{\mathrm{d}s}+\mathrm{i}k_1(s)-\frac{1}{ei(s)[k_1^2(s)+k_2^2(s)]}\left\{\frac{\mathrm{d}}{\mathrm{d}s}(ei(s)k_1^2(s))+\frac{1}{2}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}\right\}, \qquad \text{(A.6)}$$

$$M_{21}(s)=\frac{1}{2\,k_1(s)}\frac{\mathrm{d}k_1}{\mathrm{d}s}+\frac{1}{2\,ei(s)\,[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}, \qquad \text{(A.7)}$$

$$\begin{aligned}M_{31}(s)=\frac{k_1(s)}{2ei\,k_2(s)[k_1^2(s)+k_2^2(s)]}&\left\{2ei(s)(k_2(s)+\mathrm{i}\,k_1(s))\frac{\mathrm{d}k_1}{\mathrm{d}s}\right.\\&\left.+k_1(s)(k_2(s)+\mathrm{i}k_1(s))\frac{\mathrm{d}ei}{\mathrm{d}s}+\mathrm{i}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}\right\},\end{aligned} \qquad \text{(A.8)}$$

$$M_{41}(s)=M_{31}^*(s), \qquad M_{12}(s)=M_{21}(s), \qquad M_{22}(s)=M_{11}^*(s), \qquad \text{(A.9–11)}$$

$$M_{32}(s)=M_{41}(s), \qquad M_{42}(s)=M_{31}(s), \qquad \text{(A.12, 13)}$$

$$\begin{aligned}M_{13}(s)=\frac{k_2(s)}{2\,ei(s)\,k_1(s)[k_1^2(s)+k_2^2(s)]}&\left\{2\,ei(s)(k_1(s)-\mathrm{i}\,k_2(s))\frac{\mathrm{d}k_2}{\mathrm{d}s}\right.\\&\left.+k_2(s)(k_1(s)-\mathrm{i}k_2(s))\frac{\mathrm{d}ei}{\mathrm{d}s}+\mathrm{i}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}\right\},\end{aligned} \qquad \text{(A.14)}$$

$$M_{23}(s) = M^*_{13}(s), \tag{A.15}$$

$$M_{33}(s) = \frac{-1}{2k_2(s)}\frac{\mathrm{d}k_2}{\mathrm{d}s} + k_2(s) - \frac{1}{ei(s)\,[k_1^2(s)+k_2^2(s)]}\left\{\frac{\mathrm{d}}{\mathrm{d}s}(ei(s)k_2^2(s)) - \frac{1}{2}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}\right\}, \tag{A.16}$$

$$M_{43}(s) = \frac{1}{2k_2(s)}\frac{\mathrm{d}k_2}{\mathrm{d}s} - \frac{1}{2\,ei(s)\,[k_1^2+k_2^2]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}, \tag{A.17}$$

$$M_{14}(s) = M^*_{13}(s), \qquad M_{24}(s) = M_{13}(s), \qquad M_{34}(s) = M_{43}(s), \tag{A.18–20}$$

$$M_{44}(s) = \frac{-1}{2k_2(s)}\frac{\mathrm{d}k_2}{\mathrm{d}s} - k_2(s) - \frac{1}{ei(s)[k_1^2(s)+k_2^2(s)]}\left\{\frac{\mathrm{d}}{\mathrm{d}s}(ei(s)k_2^2(s)) - \frac{1}{2}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}\right\}. \tag{A.21}$$

The symbol $M^*_{jk}(s)$ stands for the complex conjugate of the matrix element $M_{jk}(s)$.

APPENDIX B: ORDER OF MAGNITUDE OF THE ELEMENTS OF MATRIX $\mathbf{M}(s)$

Here, we discuss in detail the order of magnitude of the elements of matrix $\mathbf{M}(s)$ in the system of equations (15), which governs the evolution of the wave mode amplitudes along the non-uniform beam. As mentioned in section 4.2, we consider two regimes. In the first regime, we consider the limit $\theta \to 0$, where θ is defined in section 4.2 as the ratio between the restoring force due to the tensile load and the bending moment. In the second regime, we consider the order of magnitude of the restoring force due to the tensile force to be of the same or larger order of magnitude than the order of magnitude of the bending moment ($\theta \geqslant 1$). The order of magnitude of the terms that compose the elements of matrix $\mathbf{M}(s)$ are given in terms of the non-dimensional quantities v, θ, ε, ε', δ' and λ, which were defined in the first part of section 4.2.

B.1. REGIME $\theta \to 0$

First, we need to estimate the order of magnitude of the wavenumbers $k_1(s)$ and $k_2(s)$ and their derivatives, and in terms of the quantities $v(s)$ and $\theta(s)$, they are given by the equations

$$k_1(s) = \sqrt{-\frac{\theta(s)}{2} + \frac{1}{2}\sqrt{\theta^2(s) + 4v(s)}}, \tag{B.1}$$

$$k_2(s) = \sqrt{\frac{\theta(s)}{2} + \frac{1}{2}\sqrt{\theta^2(s) + 4v(s)}}, \tag{B.2}$$

$$\frac{\mathrm{d}k_1}{\mathrm{d}s} = \frac{1}{k_1(s)}\left\{-\frac{1}{2}\frac{\mathrm{d}\theta}{\mathrm{d}s} + \frac{1}{4\sqrt{\theta^2(s)+4v(s)}}\left[2\theta(s)\frac{\mathrm{d}\theta}{\mathrm{d}s} + 4\frac{\mathrm{d}v}{\mathrm{d}s}\right]\right\}, \tag{B.3}$$

$$\frac{\mathrm{d}k_2}{\mathrm{d}s} = \frac{1}{k_2(s)}\left\{\frac{1}{2}\frac{\mathrm{d}\theta}{\mathrm{d}s} + \frac{1}{4\sqrt{\theta^2(s)+4v(s)}}\left[2\theta(s)\frac{\mathrm{d}\theta}{\mathrm{d}s} + 4\frac{\mathrm{d}v}{\mathrm{d}s}\right]\right\}. \tag{B.4}$$

Then, we discuss the order of magnitude of the elements of the matrix **M**(s). As noted in section 4.2, we need only to discuss the order of magnitude of the basic elements of the matrix **M**(s). The equations for these seven elements are given in Appendix A.

In this regime, according to equations (B.1) and (B.2), the order of magnitude of the wavenumbers are given as

$$k_1(s) \sim k_2(s) \sim O(v^{1/4}), \qquad k_1^2(s) + k_2^2(s) \sim O(v^{1/2}). \tag{B.5, 6}$$

To estimate the order of magnitude of the derivatives of the wavenumbers, we need the order of magnitude of the derivative of the quantities $v(s)$ and $\theta(s)$. According to equations (18) and (19), we have

$$\frac{\mathrm{d}v}{\mathrm{d}s} \sim \underbrace{\frac{v(s)}{m(s)}\frac{\mathrm{d}m}{\mathrm{d}s}}_{O(v\varepsilon\Lambda)} - \underbrace{\frac{v(s)}{ei(s)}\frac{\mathrm{d}ei}{\mathrm{d}s}}_{O(v\varepsilon'\Lambda)} \sim O(\max\{v\varepsilon\Lambda,\, v\varepsilon'\Lambda\}), \tag{B.7}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} \sim \underbrace{\frac{1}{ei(s)}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}}_{O(\delta\Lambda)} - \underbrace{\frac{\theta(s)}{ei(s)}\frac{\mathrm{d}ei}{\mathrm{d}s}}_{O(\theta\varepsilon'\Lambda)} \sim O(\max\{\delta\Lambda,\, \theta\varepsilon'\Lambda\}). \tag{B.8}$$

According to equations (B.3), (B.4) and the estimates above, the estimates for the order of magnitude of the derivatives of the wavenumbers are given by the equation

$$\frac{\mathrm{d}k_1}{\mathrm{d}s} \sim \frac{\mathrm{d}k_2}{\mathrm{d}s} \sim O(\max\{v^{1/4}\varepsilon\Lambda,\, v^{1/4}\varepsilon'\Lambda,\, v^{-1/4}\delta\Lambda\}). \tag{B.9}$$

The analysis of the order of magnitude of the basic elements of matrix **M**(s) follows:

$$M_{11}(s) = -\underbrace{\frac{1}{2k_1(s)}\frac{\mathrm{d}k_1}{\mathrm{d}s}}_{O(\max\{\varepsilon\Lambda,\varepsilon'\Lambda,v^{-1/2}\delta\Lambda\})} + \underbrace{\mathrm{i}k_1(s)}_{O(v^{1/4})} - \underbrace{\frac{2k_1(s)}{[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}k_1}{\mathrm{d}s}}_{O(\max\{\varepsilon\Lambda,\varepsilon'\Lambda,v^{-1/2}\delta\Lambda\})}$$

$$- \underbrace{\frac{k_1^2(s)}{ei(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}ei_1}{\mathrm{d}s}}_{O(\varepsilon'\Lambda)} + \underbrace{\frac{1}{2ei(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}}_{O(v^{-1/2}\delta\Lambda)}, \tag{B.10}$$

$$M_{21}(s) = \underbrace{\frac{1}{2k_1(s)}\frac{\mathrm{d}k_1}{\mathrm{d}s}}_{O(\max\{\varepsilon\Lambda,\varepsilon'\Lambda,v^{-1/2}\delta\Lambda\})} + \underbrace{\frac{1}{2ei(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}}_{O(v^{-1/2}\delta\Lambda)}, \tag{B.11}$$

$$M_{31}(s) = \underbrace{\frac{k_1(s)(k_2(s)+\mathrm{i}k_1(s))}{k_2[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}k_1}{\mathrm{d}s}}_{O(\max\{\varepsilon'\Lambda,\varepsilon\Lambda,v^{-1/2}\delta\Lambda\})} + \underbrace{\frac{k_1(s)(k_1(s)k_2(s)+\mathrm{i}k_1^2(s))}{2ei(s)k_2[k_1^2+k_2^2]}\frac{\mathrm{d}ei}{\mathrm{d}s}}_{O(v^{-1/4}\varepsilon'\Lambda)}$$

$$+ \frac{\mathrm{i}}{2}\underbrace{\frac{k_1(s)}{ei(s)k_2(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}}_{O(v^{-1/2}\delta\Lambda)}, \tag{B.12}$$

$$M_{13}(s) = \underbrace{\frac{k_2(s)(k_1(s) - \mathrm{i}k_2(s))}{k_1(s)[k_1^2(s) + k_2^2(s)]}\frac{\mathrm{d}k_2}{\mathrm{d}s}}_{O(\max\{\varepsilon'\Lambda, \varepsilon\Lambda, \nu^{-1/4}\delta\Lambda\})} + \underbrace{\frac{k_2(s)(k_1(s)k_2(s) - \mathrm{i}k_2^2(s))}{ei(s)k_1(s)[k_1^2(s) + k_2^2(s)]}\frac{\mathrm{d}ei}{\mathrm{d}s}}_{O(\varepsilon'\Lambda)}$$

$$+ \frac{\mathrm{i}}{2}\underbrace{\frac{k_2(s)}{ei(s)k_1(s)[k_1^2(s) + k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}(s)}_{O(\nu^{-1/2}\delta\Lambda)}, \tag{B.13}$$

$$M_{33}(s) = \underbrace{\frac{-1}{2k_2(s)}\frac{\mathrm{d}k_2}{\mathrm{d}s}}_{O(\max\{\varepsilon'\Lambda, \varepsilon\Lambda, \nu^{-1/2}\delta\Lambda\})} + \underbrace{k_2(s)}_{O(\nu^{1/4})} - \underbrace{\frac{k_2(s)}{[k_1^2(s) + k_2^2(s)]}\frac{\mathrm{d}k_2}{\mathrm{d}s}}_{O(\max\{\varepsilon'\Lambda, \varepsilon\Lambda, \nu^{-1/2}\delta\Lambda\})}$$

$$- \underbrace{\frac{k_2^2(s)}{ei(s)[k_1^2(s) + k_2^2(s)]}\frac{\mathrm{d}ei}{\mathrm{d}s}}_{O(\varepsilon'\Lambda)} + \underbrace{\frac{1}{2ei(s)[k_1^2(s) + k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}}_{O(\nu^{-1/2}\delta\Lambda)}, \tag{B.14}$$

$$M_{43}(s) = \underbrace{\frac{1}{2k_2(s)}\frac{\mathrm{d}k_2}{\mathrm{d}s}}_{O(\max\{\varepsilon'\Lambda, \varepsilon\Lambda, \nu^{-1/2}\delta\Lambda\})} + \underbrace{\frac{1}{2ei(s)[k_1^2(s) + k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}}_{O(\nu^{-1/2}\delta\Lambda)} \tag{B.15}$$

$$M_{44}(s) = \underbrace{\frac{-1}{2k_2(s)}\frac{\mathrm{d}k_2}{\mathrm{d}s}}_{O(\max\{\varepsilon'\Lambda, \varepsilon\Lambda, \nu^{-1/2}\delta\Lambda\})} - \underbrace{k_2(s)}_{O(\nu^{1/4})} - \underbrace{\frac{k_2(s)}{[k_1^2(s) + k_2^2(s)]}\frac{\mathrm{d}k_2}{\mathrm{d}s}}_{O(\max\{\varepsilon'\Lambda, \varepsilon\Lambda, \nu^{-1/2}\delta\Lambda\})}$$

$$- \underbrace{\frac{k_2^2(s)}{ei(s)[k_1^2(s) + k_2^2(s)]}\frac{\mathrm{d}ei}{\mathrm{d}s}}_{O(\varepsilon'\Lambda)} + \underbrace{\frac{1}{2ei(s)[k_1^2(s) + k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}}_{O(\nu^{-1/2}\delta\Lambda)}. \tag{B.16}$$

The order of magnitude of one element $M_{jk}(s)$ above is given by the order of magnitude of the term with the largest order of magnitude among the components of the element in question, which result in equations (29)–(35).

B.2. REGIME $\theta \geqslant 1$

We proceed in the same way as in the section above. First, we estimate the order of magnitude of the wavenumbers $k_1(s)$ and $k_2(s)$ and their derivatives. Then, we analyze the order of magnitude of the basic elements of matrix $\mathbf{M}(s)$.

In this regime, according to equations (B.1) and (B.2), the order of magnitude of the wavenumbers is given by

$$k_1(s) \sim O(\nu^{1/4}), \qquad k_2(s) \sim O(\theta^{1/2}), \tag{B.17, 18}$$

and

$$k_1^2(s) + k_2^2(s) \sim O(\theta). \tag{B.19}$$

To estimate the order of magnitude of the derivatives of the wavenumbers, we need the order of magnitude of the derivative of the quantities $v(s)$ and $\theta(s)$. The order of magnitude of these quantities are exactly the same as obtained in the previous section, and are given by equations (B.7) and (B.8).

According to equations (B.3), (B.4) and the estimates for the derivative of the quantities $v(s)$ and $\theta(s)$, the estimates for the order of magnitude of the derivatives of the wavenumbers are given by the equations

$$\frac{\mathrm{d}k_1}{\mathrm{d}s} \sim O(\max\{v^{-1/4}\delta\Lambda,\ v^{-1/4}\theta\varepsilon'\Lambda,\ \theta^{-1}v^{3/4}\varepsilon\Lambda\}), \tag{B.20}$$

$$\frac{\mathrm{d}k_2}{\mathrm{d}s} \sim O(\max\{\theta^{-1/2}\delta\Lambda,\ \theta^{1/2}\varepsilon'\Lambda,\ \theta^{-3/2}v\varepsilon\Lambda\}). \tag{B.21}$$

The analysis of the order of magnitude of the seven elements $M_{11}(s)$, $M_{21}(s)$, $M_{31}(s)$, $M_{13}(s)$, $M_{33}(s)$, $M_{43}(s)$ and $M_{44}(s)$ follows:

$$\begin{aligned} M_{11}(s) = {} & -\underbrace{\frac{1}{2k_1(s)}\frac{\mathrm{d}k_1}{\mathrm{d}s}}_{O(\max\{v^{-1/2}\delta\Lambda,v^{-1/2}\theta\varepsilon'\Lambda,\theta^{-1}v^{1/2}\varepsilon\Lambda\})} + \underbrace{\mathrm{i}k_1(s)}_{O(v^{1/4})} - \underbrace{\frac{2k_1(s)}{[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}k_1}{\mathrm{d}s}}_{O(\max\{\theta^{-1}\delta\Lambda,\varepsilon'\Lambda,\theta^{-2}v\varepsilon\Lambda\})} \\ & - \underbrace{\frac{k_1^2(s)}{ei(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}ei_1}{\mathrm{d}s}}_{O(\theta v^{1/2}\varepsilon'\Lambda)} + \underbrace{\frac{1}{2ei(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}}_{O(\theta^{-1}\delta\Lambda)}, \end{aligned} \tag{B.22}$$

$$M_{21}(s) = \underbrace{\frac{1}{2k_1(s)}\frac{\mathrm{d}k_1}{\mathrm{d}s}}_{O(\max\{v^{-1/2}\delta\Lambda,v^{-1/2}\theta\varepsilon'\Lambda,\theta^{-1}v^{1/2}\varepsilon\Lambda\})} + \underbrace{\frac{1}{2ei(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}}_{O(\theta^{-1}\delta\Lambda)}, \tag{B.23}$$

$$\begin{aligned} M_{31}(s) = {} & \underbrace{\frac{k_1(s)(k_2(s)+\mathrm{i}k_1(s))}{k_2(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}k_1}{\mathrm{d}s}}_{O(\max\{\theta^{-1}\delta\Lambda,\varepsilon'\Lambda,\theta^{-2}v\varepsilon\Lambda\})} + \underbrace{\frac{k_1(s)(k_1(s)k_2(s)+\mathrm{i}k_1^2(s))}{2ei(s)k_2(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}ei}{\mathrm{d}s}}_{O(\theta^{-1}v^{1/4}\varepsilon'\Lambda)} \\ & + \frac{\mathrm{i}}{2}\underbrace{\frac{k_1(s)}{2ei(s)k_2(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}(s)}_{O(v^{1/4}\delta\Lambda\theta^{-3/2})}, \end{aligned} \tag{B.24}$$

$$\begin{aligned} M_{13}(s) = {} & \underbrace{\frac{k_2(s)(k_1(s)-\mathrm{i}k_2(s))}{k_1(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}k_2}{\mathrm{d}s}}_{O(\max\{v^{-1/4}\theta^{-1/2}\delta\Lambda,v^{-1/4}\theta^{1/2}\varepsilon'\Lambda,v^{3/4}\theta^{-3/2}\varepsilon\Lambda\})} + \underbrace{\frac{k_2(s)(k_1(s)k_2(s)-\mathrm{i}k_2^2(s))}{ei(s)k_1(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}ei}{\mathrm{d}s}}_{O(\theta^{1/2}v^{-1/4}\varepsilon'\Lambda)} \\ & + \mathrm{i}\underbrace{\frac{k_2(s)}{2ei(s)k_1(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}}_{O(\theta^{-1/2}v^{-1/4}\delta\Lambda)}, \end{aligned} \tag{B.25}$$

$$M_{33}(s) = \underbrace{\frac{-1}{2k_2(s)}\frac{\mathrm{d}k_2}{\mathrm{d}s}}_{O(\max\{\theta^{-1}\delta\Lambda,\varepsilon'\Lambda,\theta^{-2}v\varepsilon\Lambda\})} + \underbrace{k_2(s)}_{O(\theta^{1/2})} - \underbrace{\frac{k_2(s)}{[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}k_2}{\mathrm{d}s}}_{O(\max\{\theta^{-1}\delta\Lambda,\varepsilon'\Lambda,\theta^{-2}v\varepsilon\Lambda\})}$$

$$- \underbrace{\frac{k_2^2(s)}{ei(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}ei}{\mathrm{d}s}}_{O(\varepsilon'\Lambda)} + \underbrace{\frac{1}{2ei(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}}_{O(\theta^{-1}\delta\Lambda)}, \tag{B.26}$$

$$M_{43}(s) = \underbrace{\frac{1}{2k_2(s)}\frac{\mathrm{d}k_2}{\mathrm{d}s}}_{O(\max\{\theta^{-1}\delta\Lambda,\varepsilon'\Lambda,\theta^{-2}\nu\varepsilon\Lambda\})} + \underbrace{\frac{1}{2ei(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}}_{O(\theta^{-1}\delta\Lambda)}, \tag{B.27}$$

$$M_{44}(s) = \underbrace{\frac{-1}{2k_2(s)}\frac{\mathrm{d}k_2}{\mathrm{d}s}}_{O(\max\{\theta^{-1}\delta\Lambda,\varepsilon'\Lambda,\theta^{-2}\nu\varepsilon\Lambda\})} - \underbrace{k_2(s)}_{O(\theta^{1/2})} - \underbrace{\frac{k_2(s)}{[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}k_2}{\mathrm{d}s}}_{O(\max\{\theta^{-1}\delta\Lambda,\varepsilon'\Lambda,\theta^{-2}\nu\varepsilon\Lambda\})}$$

$$- \underbrace{\frac{k_2^2(s)}{ei(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}ei}{\mathrm{d}s}}_{O(\varepsilon'\Lambda)} + \underbrace{\frac{1}{2ei(s)[k_1^2(s)+k_2^2(s)]}\frac{\mathrm{d}\bar{P}}{\mathrm{d}s}}_{O(\theta^{-1}\delta\Lambda)}. \tag{B.28}$$

The order of magnitude of the elements $M_{jk}(s)$ above is given by the order of magnitude of the component with the largest order of magnitude among the components of the element in question, which result in equations (37)–(43).

APPENDIX C: DECOUPLING OF THE PROPAGATING AND EVANESCENT PART OF THE WAVEFIELD

Here, we show that under the restriction of the tensile force and the product of the mass per unit length by the flexural rigidity being constants, the matrix elements $M_{jk}(s)$ and $M_{kj}(s)\,(j=1,2$ and $k=3,4)$ are zero.

Under the restrictions mentioned above, the wavenumbers $k_1(s)$ and $k_2(s)$ are given by the equations

$$k_1(s) = \frac{1}{ei(s)^{1/2}}\left\{-\frac{\bar{P}}{2}+\frac{1}{2}\sqrt{\bar{P}^2+4\omega^2C}\right\}^{1/2}, \tag{C.1}$$

$$k_2(s) = \frac{1}{ei(s)^{1/2}}\left\{\frac{\bar{P}}{2}+\frac{1}{2}\sqrt{\bar{P}^2+4\omega^2C}\right\}^{1/2}. \tag{C.2}$$

The derivatives of the wavenumbers can be obtained from equations (C.1) and (C.2).

C is the value of $m(s)ei(s)$ when this product is restricted to a constant value. In other words,

$$m(s)\,ei\,(s) = C. \tag{C.3}$$

Based on the expressions above for the wavenumbers $k_1(s)$ and $k_2(s)$, we realize that

$$2\,ei(s)\,(k_2(s)+\mathrm{i}k_1(s))\frac{\mathrm{d}k_1}{\mathrm{d}s} + (k_1(s)k_2(s)+\mathrm{i}\,k_1^2(s))\frac{\mathrm{d}ei}{\mathrm{d}s} = 0, \tag{C.4}$$

$$2\,ei(s)\,(k_2(s)-\mathrm{i}(k_1(s))\frac{\mathrm{d}k_1}{\mathrm{d}s} + (k_1(s)k_2(s)-\mathrm{i}\,k_1^2(s))\frac{\mathrm{d}ei}{\mathrm{d}s} = 0. \tag{C.5}$$

Equations (C.4) and (C.5) plus the restriction of constant non-dimensional tensile force imply that the elements $M_{jk}(s)$ and $M_{kj}(s)(j = 1, 2$ and $k = 3, 4)$ are zero, as we can see through their expressions in Appendix A.

In this case, the second order governing equation (50) assumes the Helmholtz-like form (65).

APPENDIX D: SECOND ORDER EQUATION WITH ANALYTIC SOLUTION

The second order governing equation assumes the form given by equation (65). The index of refraction $\tilde{n}(s)$ is given by

$$\tilde{n}(s) = \left\{ S - N \frac{\exp(k_g \lambda (s + s_0))}{1 + \exp(k_g \lambda (s + s_0))} - 4M \frac{\exp(k_g \lambda (s + s_0))}{(1 + \exp(k_g \lambda (s + s_0)))^2} \right\}^{1/2}. \tag{D.1}$$

The solution of the second order governing equation (65) for this index of refraction is given in terms of hyper-geometric functions in the form

$$\chi(s) = \frac{1}{\sqrt{k_g \lambda \exp(k_g \lambda (s - s_0))}} (1 + \exp(k_g \lambda (s - s_0)))^{\delta + \beta - \gamma + 1} 2(D F_5 + B F_6), \tag{D.2}$$

with F_5 and F_6 as the hyper-geometric functions:

$$F_5 = z^{-\delta} F(\delta, \delta - \gamma + 1, \delta - \beta + 1; 1/z), \tag{D.3}$$

$$F_6 = z^{-\beta} F(\beta, \beta - \gamma + 1, \beta - \delta + 1; 1/z). \tag{D.4}$$

The constants δ, β and γ are given in terms of the constants $\bar{\Omega}(\omega, \bar{P}, 1)$, $k_g \lambda$, S, N and M according to equations (81), (82) and (83).

The hyper-geometric functions $F_5(s)$ and $F_6(s)$ (see Brekhovskikh [11], pages 54 and 55) were used in solution (D.2) due to the chosen radiation conditions of left wave incidence and due to the sign of the parameter k_g, which is positive according to the caption of Figure 5. The reflection and transmission coefficients are given in terms of Gamma functions, according to equations (79) and (80).

APPENDIX E: REFLECTION AND TRANSMISSION COEFFICIENTS FOR THE BERNOULLI–EULER BEAM

We consider an Euler–Bernoulli beam of three pieces. At each piece, the material and geometrical properties are constants. There is a finite piece bounded by two semi-infinite pieces with the same material and geometrical properties. The material and geometrical properties for the finite piece differ from those of the two semi-infinite pieces. The governing equation for wave propagation along the beam is given by equation (6). The non-dimensional functions representing the mass per unit length and the flexural rigidity are, respectively, the functions $m(s)$ and $ei(s)$, given in this case by

$$m(s) = \begin{cases} 1 & \text{for } s < 0 \text{ and } s > \bar{L}, \\ m_1 & \text{for } 0 < s < \bar{L}, \end{cases} \tag{E.1}$$

$$ei(s) = \begin{cases} 1 & \text{for } s < 0 \text{ and } s > \bar{L}, \\ ei_1 & \text{for } 0 < s < \bar{L}. \end{cases} \tag{E.2}$$

The general solution for the governing equation (6) is given by

$$y(s)=\begin{cases} A_1\mathrm{e}^{\mathrm{i}k_1 s}+B_1\mathrm{e}^{-\mathrm{i}k_1 s}+C_1\mathrm{e}^{k_2 s}+D_1\mathrm{e}^{-k_2 s} & \text{for } s<0,\\ A_2\mathrm{e}^{\mathrm{i}k_3 s}+B_2\mathrm{e}^{-\mathrm{i}k_3 s}+C_2\mathrm{e}^{k_4 s}+D_2\mathrm{e}^{-k_4 s} & \text{for } 0\leqslant s\geqslant \bar{L},\\ A_3\mathrm{e}^{\mathrm{i}k_1(s-\bar{L})}+B_3\mathrm{e}^{-\mathrm{i}k_1(s-\bar{L})}+C_3\mathrm{e}^{k_2(s-\bar{L})}+D_3\mathrm{e}^{-k_2(s-\bar{L})} & \text{for } s>\bar{L}.\end{cases} \tag{E.3}$$

The wave numbers k_1 and k_3 are given by equation (10), and the wave numbers k_2 and k_4 are given by equation (11). $ei(s)$ and $m(s)$ in equations (10) and (11) are substituted by their values given, respectively, by equations (E.1) and (E.2). At the discontinuity of the coefficients of the governing equation (6), we have the matching conditions

$$y(\eta^+)=y(\eta^-),\qquad y_s(\eta^+)=y_s(\eta^-), \tag{E.4, 5}$$

$$ei(\eta^+)y_{ss}(\eta^+)=ei(\eta^-)y_{ss}(\eta^-), \tag{E.6}$$

$$-\bar{P}y_s(\eta^+)+ei(\eta^+)y_{sss}(\eta^+)=-\bar{P}y_s(\eta^-)+ei(\eta^-)y_{sss}(\eta^-), \tag{E.7}$$

where $\eta=0$ and $\eta=\bar{L}$. If we substitute the general solution given by equation (E.3) in the matching conditions (E.4)–(E.7) for $\eta=0$ and $\bar{L}$, we obtain the following set of systems of equations for the wave modes amplitudes along the beam:

$$[\mathbf{M}(k_1,k_2,ei(0^-))]\begin{Bmatrix}A_1\\B_1\\C_1\\D_1\end{Bmatrix}=[\mathbf{M}(k_3,k_4,ei(0^+))]\,[\mathbf{R}(k_3,k_4,0)]\begin{Bmatrix}A_2\\B_2\\C_2\\D_2\end{Bmatrix}\ \text{at } \eta=0, \tag{E.8}$$

$$[\mathbf{M}(k_3,k_4,ei(\bar{L}^-))]\,[\mathbf{R}(k_3,k_4,\bar{L})]\begin{Bmatrix}A_2\\B_2\\C_2\\D_2\end{Bmatrix}=[\mathbf{M}(k_1,k_2,ei(\bar{L}^+))]\begin{Bmatrix}A_3\\B_3\\C_3\\D_3\end{Bmatrix}\ \text{at } \eta=\bar{L}. \tag{E.9}$$

The matrices $\mathbf{M}(k_a,k_b,ei)$ and $\mathbf{R}(k_a,k_b,s)$ are given by the equations

$$[\mathbf{M}(k_a,k_b,ei(s))]=$$

$$\begin{bmatrix} 1 & 1 & 1 & 1\\ \mathrm{i}k_a & -\mathrm{i}k_a & k_b & -k_b\\ ei(s)(\mathrm{i}k_a)^2 & ei(s)(-\mathrm{i}k_a)^2 & ei(s)(k_b)^2 & ei(s)(-k_b)^2\\ -\mathrm{i}\bar{P}k_a+ei(s)(\mathrm{i}k_a)^3 & -\mathrm{i}\bar{P}k_a+ei(s)(-\mathrm{i}k_a)^3 & -\bar{P}k_b+ei(s)(k_b)^3 & \bar{P}k_b+ei(s)(-k_b)^3 \end{bmatrix}, \tag{E.10}$$

$$[\mathbf{R}(k_a,k_b,s)]=\begin{bmatrix}\exp(\mathrm{i}k_a s) & 0 & 0 & 0\\ 0 & \exp(-\mathrm{i}k_a s) & 0 & 0\\ 0 & 0 & \exp(k_b s) & 0\\ 0 & 0 & 0 & \exp(-k_b s)\end{bmatrix}. \tag{E.11}$$

Next, let us consider the radiation condition of left incidence, which is given by

$$\begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ R^- \\ C^- \\ 0 \end{Bmatrix} \text{ and } \begin{Bmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} T^- \\ 0 \\ 0 \\ D^- \end{Bmatrix}. \tag{E.12}$$

If we impose the radiation conditions (E.12) to the system of equations (E.8) and (E.9), we obtain the reflection coefficient R^- and the transmission coefficient T^- by solving numerically the resulting system of equations, which follows:

$$\begin{Bmatrix} 1 \\ R^- \\ C^- \\ 0 \end{Bmatrix} = \mathbf{M}(k_1, k_2, ei(0^-))]^{-1}[\mathbf{M}(k_3, k_4, ei(0^+))][\mathbf{R}(k_3, k_4, 0)]$$

$$[\mathbf{R}(k_3, k_4, \bar{L}]^{-1}[\mathbf{M}(k_3, k_4, ei(\bar{L}^-))]^{-1}[\mathbf{M}(k_1, k_2, ei(\bar{L}^+))] \begin{Bmatrix} T^- \\ 0 \\ 0 \\ D^- \end{Bmatrix}. \tag{E.13}$$